Matrix Algebra

$A_{m \times n}$ "rectangular array"

m rows
n columns

with typical element A_{ij}

i^{th} row
j^{th} column.

A vector is a matrix that has only 1 column or 1 row.

$$\underset{2\times 3}{A} = \begin{bmatrix} 2 & 3 \\ 6 & 2 \\ 1 & 5 \end{bmatrix} = \{ \underset{\sim}{a}_1 \ \underset{\sim}{a}_2 \}$$

$$\underset{\sim}{a}_1 = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} \quad \underset{\sim}{a}_2 = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

or $A =$ $\underset{\sim}{a}_1$

$\underset{\sim}{a}_1$ & $\underset{\sim}{a}_2$ are column vectors 3×1

<u>Square matrix</u> has same # of rows as columns i.e., $m = n$

If a matrix is <u>symmetric</u> $A_{ij} = A_{ji}$ for all i, j. NOTE, only possible for square matrices.

A <u>diagonal matrix</u> has zeros on all elements of A_{ij} $i \neq j$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The princip<u>al</u> diagonal is the nonzero entries A_{ij} $i = j$

If entries above the diagonal are zero

$$A_1 = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 7 & 3 & 2 \end{pmatrix}$$ lower triangular

A transpose of a matrix interchanges rows & columns

$$\underset{2\times 3}{B} = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 7 & 9 \end{pmatrix} \qquad \underset{3\times 2}{B^T} = \begin{pmatrix} 2 & 5 \\ 3 & 7 \\ 1 & 9 \end{pmatrix}$$

Symmetric matrices $A = A^T$

Operations

~~Ad~~ Matrices can be added or subtracted only if they are conformable. They must have the same row & column dimensions.

$A + B$ has typical element $A_{ij} + B_{ij}$

$A - B$ " " " $A_{ij} - B_{ij}$

$$A = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix} \qquad A + B = \begin{pmatrix} 3+2 \\ 2+7 \\ 1+5 \end{pmatrix}$$

Multiplication

This involves what is called <u>scalar product</u> or <u>inner product</u>.

Take 2 n vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$$\underset{n \times 1}{\underset{\sim}{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} \qquad \underset{n \times 1}{\underset{\sim}{b}} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The scalar product is denoted

$$\underset{\sim}{a}^T \underset{\sim}{b} = \sum_{i=1}^{n} a_i b_i$$

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Two Matrices

$$A_{n \times m}$$

$$B_{m \times p}$$

$$\underset{n \times m}{A} \cdot \underset{m \times p}{B} = \underset{n \times p}{C}$$

Take scalar product of each row of A times each column of B

The column dimension of the first matrix MUST = row dim of B.

$$C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 1 & 7 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 2 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$$

$2(2) + 3(3) + 3(-2)$

$$A \cdot B = \begin{matrix} 2(-1) + 3(0) + 3(1) \\ 1(-1) + 7(0) + 2(1) \end{matrix}$$

$1(2) + 7(3) + 2(-2)$

Given these rules it is generally true that

$$A \cdot B \neq B \cdot A$$

Identity

$$\underset{m \times n}{A}$$

$$A \cdot I_n = A$$

$$I_m \cdot A = A.$$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ 0 & \cdots & & 0 & 1 \end{bmatrix}$$

I is a diagonal matrix with 1 along principal diag.

One vector

$$\underset{T \times 1}{\underset{\sim}{i}_T} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$T \times 1$ vector of 1's

note $\underset{\sim}{i}_n^T \underset{\sim}{b} = \sum_{i=1}^{n} b_i$

Properties

$$A(B+C) = AB + AC$$ distributive wrt addit.

$$A + B + C = (A+B) + C = A + (B+C)$$

$$(AB)C = A(BC)$$ Associative.

$$(AB)^T = B^T A^T$$

$$(ABC)^T = C^T B^T A^T$$

$\underset{m\times n}{A}$ then $\underset{m\times m}{A^T}\underset{m\times n}{A}$ is symmetric

If symmetric $A^T A = (A^T A)^T$

$$= A^T (A^T)^T = A^T A$$

Also $AA^T = (AA^T)^T = AA^T$

Scalar mult.

$$\alpha \cdot \underset{m\times n}{A} = \begin{matrix} \alpha A_{11} & \alpha A_{12} & \cdots & \alpha A_{1n} \\ \vdots & & & \\ \alpha A_{m1} & - & - & \alpha A_{mn} \end{matrix}$$

$$\Rightarrow \alpha \cdot A = A \cdot \alpha$$

Element By Element mult. (direct product

A & B must be same dimension

then the typical element is

$A_{ij} B_{ij}$

<u>Inverse</u>

$$A \cdot A^{-1} = I \qquad A^{-1} A = I.$$

A must be square and invertible.

If A is symmetric, so is A^{-1}

If A is triangular, so is A^{-1}

If A is diagonal, so is A^{-1}

If a square matrix $A_{n \times n}$ is invertible, then its Rank is n. <u>Full Rank</u>.

If its is not invertible it is singular and has Rank less than n.

For nonsquare matrices, the rank is the largest number, m for which an $m \times m$ nonsingular matrix can be found by omitting some rows or columns.

Suppose $X_{T \times K}$ $K \leq T$

Biggest square matrix is K dim.
If Rank is K then ~~it~~.

$X^T X_{K \times K}$ has $\text{Rank}(X^T X) = K$.

And its inverse will exist.

Regression Model in Matrix form

$$y_1 = \beta_1 + \beta_2 x_1 + u_1$$

$$y_2 = \beta_1 + \beta_2 x_2 + u_2$$

$$\vdots$$

$$y_n = \beta_1 + \beta_2 x_n + u_n$$

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{pmatrix} \qquad \underset{\sim}{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\underset{\sim}{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{u}$$

y = Regressand or dependent var

X = Regressors.

If you have k regress

$$\underset{n \times k}{X} = [\,\underset{\sim}{i}_n \quad \underset{\sim}{x}_2 \quad \underset{\sim}{x}_3 \quad \cdots \quad \underset{\sim}{x}_k\,]$$

$$= \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1k} \\ 1 & x_{22} & x_{23} & \cdots & x_{2k} \\ \vdots & \vdots & & & \\ 1 & x_{n2} & & \cdots & x_{nk} \end{pmatrix}$$

$$\underset{\sim}{\beta} = \begin{matrix} \beta_1 \\ \vdots \\ \beta_k \end{matrix}$$

MOM

Basic MOM estimation estimates parameters by equating population moments to sample moments.

$$E(Y) = \mu_y = \frac{1}{n}\sum_{i=1}^{n} y_i$$

Regression

$$\underset{\sim}{y} = \underset{\sim}{X}\beta + \underset{\sim}{u} \qquad \underset{1\times K}{x_t^T} \quad \text{or} \quad y_t = x_t^T\beta + u_t$$

$$E(u_t \mid x_t) = 0 \Rightarrow \text{ by LIE} \qquad t = 1, \ldots, n$$

$$E(x_t^T u_t) = 0$$

$$\frac{1}{n}\sum_{t=1}^{n} \underset{K\times 1}{x_t}(\underset{1\times 1}{y_t} - \underset{1\times 1}{x_t^T\beta}) = 0$$

$$\sum x_t y_t - \sum x_t x_t^T \beta = 0$$

$$\sum x_t x_t^T \beta = \sum x_t y_t$$

$$\hat{\beta} = \Big(\sum_{t=1}^{n} \underset{K\times 1}{x_t} \underset{1\times K}{x_t^T}\Big)^{-1} \sum_{t=1}^{n} \underset{K\times 1}{x_t} \underset{1\times 1}{y_t}$$

$$\hat{\beta} = (X^T X)^{-1} X^T \underset{\sim}{y}$$

Least Squares

$$\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{e} \qquad \text{or}$$

$$y_t = x_t^T\beta + u_t \qquad t = 1, \dots, n$$

$$\sum_{t=1}^{T} u_t^2 = \sum_{t=1}^{T} (y_t - x_t^T \underset{-}{\beta})^2$$

$$SSR(\underset{\sim}{u}) = \underset{\sim}{u}'\underset{\sim}{u} = (\underset{\sim}{y} - X\underset{\sim}{\beta})^T(y - X\underset{\sim}{\beta})$$

$$= \underset{\sim}{y}'\underset{\sim}{y} - \underset{\sim}{y}^T X\beta - \beta^T X^T \underset{\sim}{y} + \beta' X' X\beta$$

$$\frac{\partial S}{\partial \beta} = X^T\underset{\sim}{y} - X^T y + 2X'X\beta$$

$$= -2X^Ty + 2X^TX\underset{-}{\beta} = 0$$

$$X^T X\hat{\beta} = X^T y$$

$$(X^TX)^{-1}X^Ty = \hat{\beta} \qquad \text{OLS}$$

Let $y = f(x_1, \ldots, x_n)$ or $y = f(\underset{\sim}{x})$ $\quad \underset{\sim}{x}_{n\times 1} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\frac{\partial y}{\partial \underset{\sim}{x}} = \begin{pmatrix} \partial y/\partial x_1 \\ \partial y/\partial x_2 \\ \vdots \\ \partial y/\partial x_n \end{pmatrix}_{n\times 1}$$

If $X_{n\times m}$ and $y = f(X)$

$$\frac{\partial y}{\partial X_{n\times m}} = \begin{bmatrix} \partial y/\partial x_{11} & \partial y/\partial x_{12} & \cdots & \partial y/\partial x_{1m} \\ \vdots & & & \\ \vdots & & & \\ \partial y/\partial x_{n1} & & \cdots & \partial y/\partial x_{nm} \end{bmatrix}_{n\times m}$$

Let $\underset{\sim}{y}_{m\times 1}$ and $\underset{\sim}{x}_{n\times 1}$

$$\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}^T} = \begin{pmatrix} \partial y_1/\partial x_1 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \cdots & \partial y_2/\partial x_n \\ \vdots & & \\ \partial y_m/\partial x_1 & \cdots & \partial y_m/\partial x_n \end{pmatrix}$$

$$\frac{\partial^2 \overset{\text{scalar}}{y}}{\underset{m\times 1}{\partial \underset{\sim}{x}}\ \underset{1\times n}{\partial \underset{\sim}{x}^T}} = \begin{pmatrix} \frac{\partial^2 y}{\partial x_1 \partial x_1} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_1 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 y}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 y}{\partial x_n \partial x_n} \end{pmatrix}$$

$n\times n$

$$\frac{\partial \left(\dfrac{\partial y}{\partial \underset{\sim}{x}}_{n\times 1} \right)}{\partial \underset{\sim}{x}^T} = \partial \begin{pmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{pmatrix} \cdots \Big/ \partial \underset{\sim}{x}^T$$

Let $\underset{\sim}{c}$ be a $n \times 1$ vector that <u>does not</u> depend on $\underset{\sim}{x}$

$$\underset{\sim}{z} = \underset{\substack{\sim \\ 1\times n}}{c}^T \underset{\substack{\sim \\ n\times 1}}{x}$$

$$\frac{\partial z}{\partial \underset{\sim}{x}_{n\times 1}} = \frac{\partial c^T \underset{\sim}{x}}{\partial \underset{\sim}{x}} = \frac{\partial \underset{\sim}{x}^T c}{\partial \underset{\sim}{x}} = \underset{\sim}{c}$$

z is scalar, $\underset{\sim}{x}$ is $n \times 1$ $\Rightarrow$ <u>result</u> $n \times 1$

Let C be $n \times m$, $\underset{\sim}{x}$ $n \times 1$

$$z = C' \underset{\sim}{x} \quad \begin{matrix} \cancel{n\times m}\ n\times 1 \\ m \times n \\ m\times 1 \end{matrix}$$

$$\frac{\partial z}{\partial \underset{\sim}{x}^T} = \frac{\partial C' \underset{\sim}{x}_{n\times 1}}{\partial \underset{\sim}{x}^T_{1\times n}}$$

$$\frac{\partial z^{m\times 1}}{\partial \underset{\sim}{x}^T_{1\times n}} = C^T_{m\times n} \quad \text{or} \quad \frac{\partial \underset{\sim}{x}^T C}{\partial \underset{\sim}{x}} \begin{matrix} 1\times n \ \ n\times m \\ 1 \times m \end{matrix} = C_{n\times m}$$

Let A be square

$\cdot\ z = \underset{\sim}{x}^T A\, \underset{\sim}{x}$

$$\frac{\partial z}{\partial \underset{\sim}{x}} = A^T \underset{\sim}{x} + A\underset{\sim}{x} = (A^T + A)\,\underset{\sim}{x}$$

If A symmetric $= 2A\underset{\sim}{x}$