Exact Tests in the Linear Reg Model.

$\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{u}$ $\quad E(\underset{\sim}{u}|X) = 0$ <u>in fact</u>.

$\underset{\sim}{u}|X \sim N(0, \sigma^2 I_n)$

LS. $\hat{\beta} = (X^TX)^{-1}X^Ty = \cancel{X^Tx}\ \beta + (X^TX)^{-1}X^T\underset{\sim}{u}$

using 'D' theorem.

$\tilde{\beta}|X \sim N(\beta, \sigma^2(X^TX)^{-1})$

Consider linear hypotheses about the parameters β.

H_0: $R\underset{\sim}{\beta} = \underset{\sim}{r}$

H_A $R\beta \neq \underset{\sim}{r}$

R $J\times K$ matrix of known const.

r $J\times 1$ vector of const.

$\text{Rank}(R) = J \leq K \Rightarrow$ set of J linearly indep hypotheses.

$$\hat{\beta} \mid X \sim N\left(\underset{\sim}{\beta}, \sigma^2 (X^T X)^{-1}\right)$$

$$R\hat{\beta} \mid X \sim N\left(R\beta, \sigma^2 R (X^T X)^{-1} R^T\right)$$

$$(R\hat{\beta} - R\beta) \mid X \sim N\left(0, \sigma^2 R (X^T X)^{-1} R^T\right)$$

Use Theorem

$$(R\hat{\beta} - R\beta)^T \left[\sigma^2 R (X^T X)^{-1} R^T\right]^{-1} (R\hat{\beta} - R\beta)$$

$$\sim \chi^2_J$$

Under H_0: $R\beta = r$

$$Q = \frac{(R\hat{\beta} - r)^T \left(R (X^T X)^{-1} R^T\right)^{-1} (R\hat{\beta} - r)}{\sigma^2} \sim \chi^2_J$$

NOTE: $$\hat{\sigma}^2 = \frac{\hat{u}^T\hat{u}}{n-k}$$

$$\hat{u} = M_X u \qquad u \sim N(0, \sigma^2 I_n)$$

$$\therefore \frac{\hat{u}^T\hat{u}}{n-k} = \frac{u^T M_X u}{n-k} \sim \sigma^2 \frac{\chi^2_{n-k}}{n-k}$$

$$\text{or} \quad \hat{\sigma}^2/\sigma^2 \sim \chi^2_{n-k}/n-k$$

and indep of Q

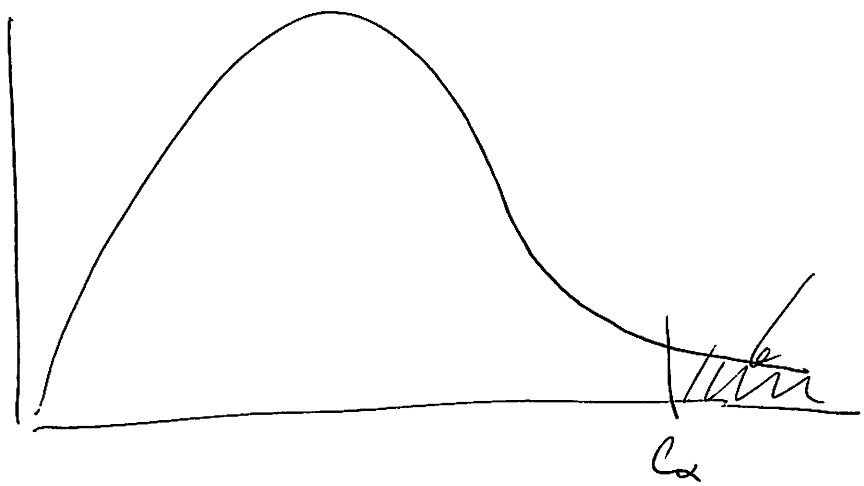
So, $$Q/J \sim \chi^2_J/J$$

Let $$F = \frac{Q/J}{\hat{\sigma}^2/\sigma^2} \sim \frac{\chi^2_J/J}{\chi^2_{n-k}/n-k} \sim F_{J,n-k}$$

$$\lambda_1 = F = \frac{(R\hat{\beta}-r)^T (R(X^TX)^{-1}R^T)^{-1}(R\hat{\beta}-r)/J}{\hat{\sigma}^2} \sim F_{J,n-k}$$

if $R\beta = r$ i.e. H_0 True.

let $\alpha = .05$



If $F \geq C_\alpha$ reject.

This is often referred to as a Wald type Test. Based on Quadratic forms in Normal R.V.

Example:

$$y = A L^{\alpha_1} K^{\alpha_2} \quad \text{Cobb Douglas.}$$

$$\alpha_1 + \alpha_2 = 1 \Rightarrow CRS.$$

~~$\ln y = A +$~~ β_1

linearize & add on error.

$$\ln y_t = \ln(A) + \alpha_1 \ln(L_t) + \alpha_2 \ln(K_t) + u_t$$

$$CRS \Rightarrow \alpha_1 + \alpha_2 = 1$$

Test

$H_0: \alpha_1 + \alpha_2 = 1$

$H_A \quad \alpha_1 + \alpha_2 \neq 1$

$$R = (0 \quad 1 \quad 1) \qquad J = 1$$

$$\underset{\sim}{r} = 1$$

$$\frac{(R\hat{\beta} - r)^T (R(X^TX)^{-1}R^T)^{-1} (R\hat{\beta} - r)/1}{\hat{\sigma}^2} \sim F_{1, n-k}$$

if H_0 True.

When there is a single linear hypothesis you can also perform a t test — which can also be 1-sided.

$H_0.\ \alpha_1 + \alpha_2 = 1$

$H_A\ \ \alpha_1 + \alpha_2 \neq 1$

$$\frac{\hat{\alpha}_1 + \hat{\alpha}_2 - 1}{\hat{SE}(\hat{\alpha}_1 + \hat{\alpha}_2 - 1)} \sim t_{n-k} \text{ if } H_0 \text{ True.}$$

$H_0.\ \beta_2 + \beta_3 = 1$ AND $\beta_4 = 3\beta_5$

STATA

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regress y x2 x3 ... xk
test (x2 + x3 = 1) (x4 - 3x5 = 0)
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$F_{2,n-k}$ if H_0 True.

Gretl:

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ols y const x2 x3 x4 ... xk
restrict
b[2]+b[3]=1
b[x4]-3*b[x5]=0
end restrict
```

In The linear Regression model there are 2 numerically equivalent forms of The stat.

~~Stat~~

Suppose we can impose the Restrictions on The model and Minimize SSE s.t. exact linear Rest (easily done) and

β^* is The restricted LS estimator.

β^* obey The rest $\Rightarrow R\beta^* = r$

$H_0: R\beta = r$
$H_A \quad R\beta \neq r$

$RSSE = (y - X\beta^*)^T (y - X\beta^*)$
$USSR = (y - X\hat{\beta})^T (y - X\hat{\beta})$

$$\lambda_2 \quad \frac{(RSSR - USSR)/J}{USSR/(n-k)} \overset{a}{\sim} F_{J, n-k} \text{ if } H_0 \text{ True.}$$

$\hat{\sigma}^2$

Also

$$\lambda_3 \quad \frac{(\hat{\beta} - \beta^*)^T X^T X (\hat{\beta} - \beta^*)/J}{\hat{\sigma}^2} \overset{a}{\sim} F_{J, n-k} \text{ if } H_0 \text{ True.}$$

A single test

For instance, test $\beta_2 = 0$ or
$\beta_2 + \beta_3 = 1$

$$R_1 \underset{\sim}{\beta} = [\,0 \;\; 1 \;\; 0 \cdots 0\,] \underset{\sim}{\beta} = r_1 = 0$$

OR

$$R_1 \underset{\sim}{\beta} = [\,0 \;\; 1 \;\; {+1} \;\; 0 \;\; 0 \cdots 0\,] \underset{\sim}{\beta} = r_1 = 1$$

scalar.

$$\lambda = \frac{(R_1\tilde{\beta} - r_1)\,[R_1(X'X)^{-1}R_1']^{-1}(R_1\tilde{\beta} - r_1)}{\hat{\sigma}^2}$$

$$= \frac{(R_1\tilde{\beta} - r_1)^2}{\hat{\sigma}^2\,[R_1(X'X)^{-1}R_1']} \sim F_{1,T-K} \text{ if } H_0 \text{ True.}$$

From normal distribution theory we know that

$$F_{1,T-K} = [t_{T-K}]^2$$

$\therefore$ taking the square root of λ yields a t statistic.

$$\frac{R_1\beta - r_1}{\sqrt{\hat{\sigma}^2(R_1(X'X)^{-1}R_1')}} \sim t_{T-K} \text{ if } H_0 \text{ True.}$$

Theory of single restriction

$$y = X_1 \beta_1 + x_2 \beta_2 + u \qquad u \sim N(0, \sigma^2 I_n)$$

$$M_1 = I - X_1 (X_1^T X_1)^{-1} X_1^T$$

use FWL

$$M_1 y = M_1 X_1 \beta_1 + M_1 x_2 \beta_2 + M_1 u$$
$$= M_1 x_2 \beta_2 + \text{res}.$$

$$\text{ols} = (x_2^T M_1 x_2)^{-1} x_2^T M_1 y \qquad \text{scalar.}$$

$$\text{Var} = \underbrace{\sigma^2 (x_2^T M_1 x_2)^{-1}}_{\text{scalar}}$$

$$\hat\beta_2 \sim N\left(\beta_2, \frac{\sigma^2}{x_2^T M_1 x_2}\right) \qquad \begin{array}{l} H_0: \beta_2 = 0 \\ H_A: \beta_2 \neq 0 \end{array}$$

$$\frac{\hat\beta_2 - \beta_2}{\sqrt{\sigma^2 / x_2^T M_1 x_2}} = \frac{x_2^T M_1 y \,/\, x_2^T M_1 x_2}{\sigma / (x_2 M_1 x_2)^{1/2}} \sim N(0,1)$$

Overall F stat

Test used to determine overall signif of the regression model.

$$y_i = \beta_1 + \beta_2 x_{t2} + \cdots + \beta_k x_{tk} + e_t$$

If $\beta_2 = \beta_3 = \cdots = \beta_k = 0$ then this reduces to model of the mean

None of the x's affect $E(y)$.

$H_0: \beta_2 = \cdots = \beta_k = 0$ $\quad\quad J = k-1$

H_A not H_0.

$$\underset{(k-1 \times k)}{R} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \\ 0 & 0 & & & 1 \end{bmatrix} = \left[\underset{\sim}{0} \,\vdots\, I_{k-1} \right]$$

$\underset{\sim}{r} = \underset{\sim}{0}$

Wald test $\overset{a}{\sim} F_{k-1,\, n-k}$ if H_0 True.

Reject if $\lambda_1 \geq C_\alpha$ from this dist or if Pval $< \alpha$.

which permits one sided tests or two sided.
confidence intervals.

Asymptotic Properties of LS

So, far we've said $e \sim N(0, \sigma^2 I_T)$
and that enabled us to say

$$\tilde{\beta} = (X'X)^{-1}X'y \sim N(\beta, \sigma^2 (X'X)^{-1})$$

what if e is NOT normally distributed?

$$y = X\beta + e \qquad e \sim (0, \sigma^2 I_T)$$

In this case we have to rely on LS asymptotic properties in order to do hypoth tests and form CI.

$$b \underset{=}{\sim} (\beta, \sigma^2 (X'X)^{-1})$$

Relaxing $E(e_t | \mathcal{X}_t) = 0$ and $E(e_t^2 | \mathcal{X}_t) = \sigma^2$

Another assumption

$$\underset{T \to \infty}{\text{plim}} \frac{X'X}{T} = Q \quad \text{finite \& Nonsingular}$$

well, it can be shown that

$$\sqrt{T}(b - \beta) \xrightarrow{d} N(0, \sigma^2 Q^{-1})$$

Then, $b \overset{asy}{\sim} N(\beta, \sigma^2 (X'X)^{-1})$
which provides the basis for nearly all inference using OLS.

Furthermore, using "D" Theorem again

$$\sqrt{T}(Rb - R\beta) \xrightarrow{d} N(0, \sigma^2 R Q^{-1} R')$$

and

$$\frac{\sqrt{T}(Rb-R\beta)' [R Q^{-1} R']^{-1} (Rb - R\beta)\sqrt{T}}{\sigma^2} \xrightarrow{d} \chi^2_J$$

Replacing $\underset{T\to\infty}{\text{plim}} \frac{X'X}{T}$ with its finite sample counterpart yield

$$\frac{(Rb-R\beta)' [R(X'X)^{-1}R']^{-1}(Rb-R\beta)}{\sigma^2} \xrightarrow{asy} \chi^2_J$$

Also, replacing σ^2 with $\hat{\sigma}^2$ does not alter the asymptotic distribution, so

$$\lambda^* = \frac{(Rb-R\beta)' [R(X'X)^{-1}R']^{-1}(Rb-R\beta)}{\hat{\sigma}^2} \overset{asy}{\sim} \chi^2_J$$

When $J = 1$

$$w = \frac{Rb - R\beta}{\hat{\sigma}\sqrt{R(X'X)^{-1}R'}} \xrightarrow{d} N(0,1)$$

In practice, people approximate these distributions with

F and t

$\frac{\lambda^*}{J} \overset{a}{\sim} F_{J,T-K}$ which is

a bit more conservative than using the χ^2_J

$J F_{J,T-K} \rightarrow \chi^2_J$!

Overall F-Test

This is a test that is often used to determine the overall ~~sig~~ statistical significance of your model.

Ex $\qquad y_t = \beta_1 + \beta_2 x_{t2} + \cdots + \beta_K x_{tK} + e_t$

if $\beta_2 = \beta_3 = \cdots = \beta_K = 0$ then your model reduced to the model of the mean. None of the x's affect y and you would conclude that you have no model.

$H_0: \beta_2 = \beta_3 = \cdots = \beta_K = 0$

$H_A \quad \beta_2 \neq 0$ or $\beta_3 \neq 0 \cdots$ or $\beta_K = 0$

$$R = \underset{K-1 \times K}{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & \vdots \\ \vdots & \vdots & 0 & & & \\ \vdots & \vdots & \vdots & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{bmatrix}} \qquad \underset{\sim}{r} = \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}$$

use Wald test $\quad J = K-1$

$\lambda \sim F_{K-1,T-K}$ if H_0 true.

RESET

A general test for specification is the RESET

The reset test will detect extreme problems of ~~either relevant omitted variables or~~ poor choice of functional form. Wooldridge shows that this test only has power in the 1st instance in special cases. ∴ Test for Func Form.

$H_0: X'\underset{\sim}{e} = 0$

$H_A: X'\underset{\sim}{e} \neq 0$

The alternative suggests a number of possibilities

(1) e includes relevant variables

(2) relationship between x & y is not linear.

(3) Wooldridge (1995) shows its really 2 NOT 1.

The test is simple to execute.

(1) Estimate the linear regression

$$y = \beta_1 + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} + e_t$$

and obtain the residuals

$$\hat{y}_t = b_1 - b_2 x_{t2} - \cdots - b_k x_{tk}.$$

(2) Form powers of $\hat{e}_t$

$$\hat{y}_t^2, \hat{y}_t^3, \text{ etc.}$$

(3) Run auxillary reg.

$$y_t = \beta_1 + \beta_2 x_{t2} + \cdots + \beta_k x_{tk} + \delta_2 \hat{y}_t^2 + \delta_3 \hat{y}_t^3 + e$$

$H_0: \delta_2 = \delta_3 = 0$

$H_A: \delta_2 \neq 0$ or $\delta_3 \neq 0$.

$\lambda \sim F_{2, T-k-2}$ if H_0 true.

if $\lambda \geq c_\alpha$ Reject and conclude misspec.

Structural Change

One of the more common uses of the F test is to test for the absence of structural change in the model's parameters

Suppose we have a regression model

$$y_t = \beta_1 + \beta_2 x_{t2} + \beta_3 x_{t3} + e_t \qquad t = 1, \ldots, T$$

and we suspect that at some point in our sample, $1 < n < T$ we suspect that the regression function changes.

$$\begin{aligned}
y_1 & \quad \beta_1 + \beta_2 x_{12} + \beta_3 x_{13} + e_1 \\
y_2 & \quad \beta_1 + \beta_2 x_{22} + \beta_3 x_{23} + e_2 \\
\vdots & \quad \vdots \\
y_n & = \beta_1 + \beta_2 x_{n2} + \beta_3 x_{n3} + e_n \\
y_{n+1} & \quad \beta_1^* + \beta_2^* x_{n+1,2} + \beta_3^* x_{n+1,3} + e_{n+1} \\
\vdots & \quad \vdots \\
y_T & \quad \beta_1^* + \beta_2^* x_{T,2} + \beta_3^* x_{T3} + e_T
\end{aligned}$$

$$\begin{matrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \end{matrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{matrix} \underset{\sim}{\beta}_1 \\ \underset{\sim}{\beta}_2 \end{matrix} + \begin{matrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \end{matrix} \qquad \begin{matrix} X_1 \ n \times K & \beta_1 \ K \times 1 \\ X_2 \ T-n \times K & \beta_2 \ K \times 1 \end{matrix}$$

NOTE: $n \geq K \quad T-n \geq K$

One test of structural change is Wald Test

$$\underset{\sim}{y} = X \underset{\sim}{\beta} + \underset{\sim}{e}$$

y $T \times 1$

X $T \times 2K$

β $2K \times 1$

e $T \times 1$

$$R = [\, I_K \, \vdots \, -I_K \,] \qquad \underset{\sim}{r} = \underset{\sim}{0}$$

Estimate b using OLS, and σ^2 using $\hat{\sigma}^2$

$$\lambda = \frac{(Rb-r)' \left[R'(X'X)^{-1}R' \right]^{-1} (Rb-r)}{\hat{\sigma}^2 K} \sim F_{K, T-2K}$$

if H_0 True.

An equivalent way is to use λ_2, the other form of the statistic, and run ~~sepp~~ separate regressions

$$\lambda_2 = \frac{(SSE_R - SSE_u)/K}{SSE_u / T-2K} \sim F_{K, T-2K} \text{ if } H_0 \text{ True.}$$

This test is sometimes referred to as the <u>Chow</u> Test.

SSE_R comes from the restricted regression

$\beta_1 = \beta_2 \implies$

Model setup.

$$y = X\beta + e$$

$$\underset{T\times k}{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{matrix} n\times k \\ T-m\times k \end{matrix} \qquad \underset{k\times 1}{\beta} =$$

$SSE_u = SSE_1 + SSE_2$ unrestricted.

$$y_1 = X_1\beta_1 + e_1 \implies b_1 \implies SSE_1$$

$$y_2 = X_2\beta_2 + e_2 \implies b_2 \implies SSE_2$$

* There are many variations of this where you can allow 1 or more of the parameters to differ across regimes.

Chow test ~~for~~ with unequal variance

One of the assumptions of the basic Chow Test is that the variance of the two regimes are the same.

Here is an (asymptotic) test.

Let $\hat{\theta}_1 \sim N(\theta_1, V_1)$ and $\hat{\theta}_2 \sim N(\theta_2, V_2)$
and $\hat{\theta}_1$ & $\hat{\theta}_2$ are stat. independent.

$$\hat{\theta}_1 - \theta_1 \sim N(0, V_1) \qquad \hat{\theta}_2 - \theta_2 \sim N(0, V_2)$$

$$(\hat{\theta}_1 - \theta_1) - (\hat{\theta}_2 - \theta_2) \sim N(0, V_1 + V_2)$$

H_0: $\theta_1 = \theta_2$
H_A $\theta_1 \neq \theta_2$

Under H_0:

$$(\hat{\theta}_1 - \hat{\theta}_2)' (V_1 + V_2)^{-1} (\hat{\theta}_1 - \hat{\theta}_2) \sim \chi^2_K$$

T-Test

$$y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + \cdots + \beta_k x_{ik} + u_i$$

$$E(\underset{\sim}{u} \mid X) = 0$$
$$E(\underset{\sim}{u}^2 \mid X) = \sigma^2 I_n$$
$$\text{Rank}(X) = k \leq n$$

OLS estimate $\hat{\beta} = (X^T X)^{-1} X^T \underset{\sim}{y}$

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{pmatrix} = (X^T X)^{-1} X^T \underset{\sim}{y} \sim N\left(\beta, \sigma^2 (X^T X)^{-1}\right)$$

Recall $\sigma^2 (X^T X)^{-1}$ is the Var Cov matrix of $\hat{\beta}$. It is estimated $\hat{\sigma}^2 (X^T X)^{-1}$ where $\hat{\sigma}^2 = \dfrac{\hat{u}^T \hat{u}}{n-k}$

and contains variances of each $\hat{\beta}_i$ on i^{th} diag and covariance $(\hat{\beta}_i, \hat{\beta}_j)$ on the i^{th}, j^{th} element.

$H_0: \hat{\beta}_i = c$

$H_A: \hat{\beta}_i > c$

$$\hat{\beta}_i \overset{a}{\sim} N\left(\beta_i, \sigma^2 (X^T X)^{-1}_{ii}\right)$$

where $\sigma^2 (X^T X)^{-1}_{ii}$ is the element in i^{th} row & i^{th} column of $\sigma^2 (X^T X)^{-1}$.

It can be shown

$$\frac{\hat{\beta}_i - \beta_i}{\sqrt{\sigma^2 (X^T X)^{-1}_{ii}}} \overset{a}{\sim} N(0,1)$$

and $\hat{\sigma}^2 / \sigma^2 \sim \chi^2_{n-k} / n-k$

and indep.

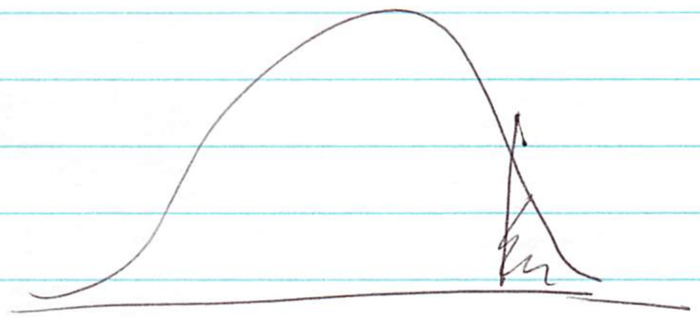
$\therefore$

$$\frac{\hat{\beta}_i - \beta_i}{\sqrt{\hat{\sigma}^2 (X^T X)^{-1}_{ii}}} \overset{a}{\sim} t_{n-k}$$

When H_0: true.

$$t = \frac{\hat{\beta}_2 - c}{\sqrt{\hat{\sigma}^2 (X'X)^{-1}_{ii}}} \sim t_{n-k}$$

$\alpha = .05$

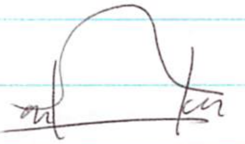


Reject if $t > t_{c, n-k}$.

where t_c is the α level critical value from t_{n-k}.

$H_0: \beta_0 = c$
$H_A \ \beta_0 \neq c$

$t \sim t_{n-k}$ if H_0 True



Reject if $|t| > t_{c, \alpha/2}$