GLS

$$y = X\beta + u \qquad E(u|X) = 0$$

$$E(uu^T) = \Omega \neq \sigma^2 I_n$$

Recall,

$$\hat{\beta} = (X^T X)^{-1} X^T y \quad - \text{LS}$$

$$\hat{\beta} \overset{a}{\sim} N\left(\beta, (X^T X)^{-1} X^T \Omega X (X^T X)^{-1}\right)$$

but is not efficient.

Let Ψ be a matrix ($n \times n$, pos def)
such that $\Psi^T u \sim (0, I_n)$

using Ψ^T to transform model

$$\Psi^T y = \Psi^T X \beta + \Psi^T u$$

$$y^* = X^* \beta + u^* \qquad u^* \text{ obeys Assump of GM. } \therefore \text{ use OLS}$$

$$\hat{\beta}_{GLS} = (X^{*T} X^*)^{-1} X^{*T} y^*$$

$$= (X^T \Psi \Psi^T X)^{-1} X^T \Psi \Psi^T y$$

Note: $u \sim (0, \Omega)$

$$\Psi^T u \sim (0, \Psi^T \Omega \Psi^{*})$$

$$\Psi^T \Omega \Psi^{*} = I$$

$$= \Psi^{T^{-1}} \Psi^T \Omega \Psi^{*} (\Psi^{*})^{-1} = \Psi^{T^{-1}} (\Psi^{*})^{-1}$$

$$= (\Psi^{*} \Psi^T)^{-1}$$

$$\Rightarrow (\Psi^{*} \Psi^T)^{-1} = \Omega \quad \text{or}$$

$$\Psi^{*} \Psi^T = \Omega^{-1}$$

$$\therefore \quad \hat{\beta}_{GLS} = (X^T \Omega^{-1} X)^{-1} X^T \Omega^{-1} y$$

$E(\hat{\beta}_{GLS}) =$

$$\cancel{Cov(\hat{\beta}_{GLS}) = E\left[\left(\hat{\beta}_{GLS} - E(\hat{\beta}_{GLS})\right)\left(\hat{\beta}_{GLS} - E(\hat{\beta}_{GLS})\right)\right]^{-1}}$$

$$\cancel{= (X^{*T} X^{*})^{-1} \sigma^2}$$

$=$

Actually this is the estimator you get if minimizing the GLS objective function

$$= (y - X\beta_{GLS})^T \Omega^{-1} (y - X\beta_{GLS}) \qquad (7.06)$$

$$= (y - X\beta_{GLS})^T \Psi\Psi^T (y - X\beta_{GLS})$$

$$= (\Psi^T y - \Psi^T X \beta_{GLS})^T (\Psi^T y - \Psi^T X \beta_{GLS})$$

$$= (y^* - X^* \beta_{GLS})^T (y^* - X^* \beta_{GLS})$$

The FOC is.

$$X^{*T} (y^* - X^* \beta_{GLS}) = 0$$

$$X^T \Psi \; (\Psi^T y - \Psi^T X \beta_{GLS}) = 0$$

$$X^T \Psi \Psi^T (y - X\beta_{GLS}) = 0$$

$$X^T \Omega^{-1} (y - X\beta_{GLS}) = 0 \qquad (7.07)$$

$$\hat{\beta} = (X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1}y$$

$$= (X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1}(X\beta + u)$$

$$= \beta + (X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1}u$$

$$E(\hat{\beta}) = \beta + E[X^T\Omega^{-1}X]^{-1}X^T\Omega^{-1}u$$

$$= \beta \quad \text{since} \quad E(u/X) = 0$$

$$E\left((\hat{\beta}-\beta)(\hat{\beta}-\beta)^T\right) =$$

$$= E\left[(X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1}u\,u^T\Omega^{-1}X(X^T\Omega^{-1}X)^{-1}\right]$$

$$= E\left[(X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1}\Omega\,\Omega^{-1}X(X^T\Omega X)^{-1}\right]$$

$$= (X^T\Omega^{-1}X)^{-1}$$

$$\therefore \quad \hat{\beta}_{GLS} \sim (\beta, (X\Omega^{-1}X)^{-1})$$

which By GM is efficient.

$$Cov(\hat{\beta}_{LS}) - Cov(\hat{\beta}_{GLS}) = \Delta$$

$$(X^TX)^{-1}X^T\Omega X(X^TX)^{-1} - (X^T\Omega^{-1}X)^{-1} = \Delta \quad \text{p.s.d.}$$

So, if Ω^{-1} is known

$$\hat{\beta}_{GLS} = (X^T \Omega^{-1} X)^{-1} X^T \Omega^{-1} y \quad \text{is efficient.}$$

Furthermore,

under general conditions $E[u|X]=0$ & $E[u_t|\mathcal{F}_t]=0$

$$\sqrt{n}(\beta_{GLS} - \beta) \xrightarrow{d} N\left(0, \text{plim}\left(\frac{X^T \Omega^{-1} X}{n}\right)^{-1}\right)$$

Of course, if Ω^{-1} known, you could transform $y = X\beta + u$ using ψ^T

where $\psi\psi^T = \Omega^{-1}$

$$\psi^T y = \psi^T X\beta + \psi^T u$$

$$y^* = X^*\beta + u^*$$

$$\beta_{GLS} \overset{a}{\sim} N(\beta, (X^T \Omega^{-1} X)^{-1})$$

Est. of Cov.

$$\hat{\sigma}_g^2 = (y - X\beta_{GLS})^T \Omega^{-1} (y - X\beta_{GLS}) / n-k$$

$$\xrightarrow{P} 1 \quad \text{and}$$

$$\widehat{\text{Cov}}(\hat{\beta}) = \hat{\sigma}_g^2 (X^{*T} X^*)^{-1} = \hat{\sigma}_g^2 (X^T \Omega^{-1} X)^{-1}$$

Hypothesis Tests

$H_0: R\beta = r$ $\quad$ R $J \times K$, $\text{Rank}(R) = J \leq K$

$H_A: R\beta \neq r$ $\quad$ r $J \times 1$

$$\lambda_1 = \frac{(R\hat{\beta}_{GLS} - r)^T [R(X^{*T}X^*)^{-1}R^T]^{-1}(R\hat{\beta}_{GLS} - r)/J}{\hat{\sigma}_g^2}$$

$\xrightarrow{\sim} \chi^2_J / J$ if H_0 True.

or approximate with $F_{J, n-k}$.

NOTE: on untransformed model

replace $X^{*T}X^* \overset{w}{=} X^T \Omega^{-1} X$.

Feasible GLS

Unfortunately Ω and Ψ are usually unknown.

$\frac{n(n+1)}{2}$ elements. in Ω

So, the idea is to estimate Ω & Ψ as a function of a small # of parameters, θ

$\Omega(\theta)$ estimated by $\Omega(\hat{\theta}) \equiv \hat{\Omega}$
$\Psi(\theta)$ " " $\Psi(\hat{\theta}) \equiv \hat{\Psi}$

and FGLS

$$\hat{\hat{\beta}}_{GLS} = (X^T\hat{\Omega}^{-1}X)^{-1}X^T\hat{\Omega}^{-1}y \text{ or}$$
$$(\hat{X}^{*T}\hat{X}^{*})^{-1}\hat{X}^{*T}\hat{y}^{*}$$

with $\hat{X}^{*} = \hat{\Psi}^T X$
$\hat{y}^{*} = \hat{\Psi}^T y$

$$\sqrt{n}\,(\hat{\hat{\beta}}_{GLS} - \beta) \xrightarrow{d} N\left(0,\ \text{plim}\left(X^T\frac{\Omega^{-1}}{n}X\right)^{-1}\right)$$

same as GLS!