Hypothesis Testing

3 classical tests

LR
LM
Wald
} asymptotically equivalent

let $\ell(\theta|y)$ be the log likelihood function and $r(\theta) = 0$ be J smooth functions of the parameters. When the parameter space is restricted according to $r(\theta) = 0$ then the MLE's satisfy them.

$$H_0: r(\theta) = 0$$
$$H_A: r(\theta) \neq 0$$

LR

$\hat{\theta}$ be restricted MLE
$\tilde{\theta}$ be the unrestricted

$$LR = 2\left(\underset{\text{un}}{\ell(\tilde{\theta})} - \underset{\text{res}}{\ell(\hat{\theta})}\right) \overset{a}{\sim} \chi^2_J \text{ if } H_0 \text{ true}$$

Wald Test

$$Var(r(\hat{\theta})) \stackrel{a}{=} \underset{J\times k}{R(\theta)}\ \underset{k\times k}{Var(\hat{\theta})}\ \underset{k\times J}{R(\theta)'}$$

where $R(\theta) = \dfrac{\partial\, r(\underset{\sim}{\theta})_{J\times 1}}{\partial \underset{\sim}{\theta}'}$

$$W = r(\hat{\theta})' \left[R(\hat{\theta})\, Var(\hat{\theta})\, R(\hat{\theta})' \right]^{-1} r(\hat{\theta})$$

$$\xrightarrow{d} \chi^2_J \text{ if } H_0 \text{ True.}$$

These tests should be used with some caution. In finite samples they are not invariant to how you write the restrictions

$$y_t = \beta_1 + \beta_2 x_{t2} + \beta_3 x_{t3} + e_t$$

$$\beta_2 \cdot \beta_3 = 1$$

$$\left.\begin{array}{ll} r_1 & \beta_2 - 1/\beta_3 = 0 \\ r_2 & \beta_3 - 1/\beta_2 = 0 \\ r_3 & \beta_2 \cdot \beta_3 - 1 = 0 \end{array}\right]$$

will yield 3 different W's!

LM Test

There are several forms for these tests, some easier to obtain than others.

As the name suggests, these are the outcomes of constrained optimization problem where the likelihood, or log-likelihood, is maximized subject to $r(\underset{\sim}{\theta}) = 0$

$$\ell(\underset{\sim}{\theta}) - \underset{1\times J}{r'(\theta)}\, \underset{J\times 1}{\underset{\sim}{\lambda}}$$

where $\underset{\sim}{\lambda}$ contains the J Lagrange multipliers

The F.O.C.

$$\underset{\text{Score}\ k\times 1}{g(\tilde{\theta})} - R'(\tilde{\theta})\, \underset{J\times 1}{\tilde{\lambda}} = 0$$

$$r(\underset{\sim}{\theta}) = 0$$

$$\underset{J\times k}{R} = \frac{\partial \underset{J\times 1}{r(\theta)}}{\partial \underset{1\times k}{\theta'}}$$

S

$$LM_1 = g'(\tilde{\theta})\,\tilde{I}^{-1}(\tilde{\theta})\,g(\tilde{\theta})$$

score form.

$$LM_2 = \tilde{\lambda}'\tilde{R}\,\tilde{I}^{-1}\tilde{R}'\tilde{\lambda}$$

where $\tilde{\lambda}, \tilde{R}, \tilde{I}$ are all evaluated at the Restricted MLE's

$LM_1, LM_2 \xrightarrow{a} \chi^2_J$ if Restrictions True.

LM and.

<u>Artificial Regressions</u>

LM_1 is Based on $I(\hat{\theta})$ and as it turns out, there are as many ways to compute this as there are valid ways to comput $I(\hat{\theta})$.

For example, computing $I(\hat{\theta})$ using BHHH' - OPG

$$\tilde{g}'(\tilde{G}'\tilde{G})^{-1}\tilde{g}$$

This # can be computed using regression analysis.

$$\underset{\sim}{\dot{d}_\pi} = \underset{n \times k}{G(\theta)} \underset{k \times 1}{\gamma} + \varepsilon_i$$

γ are unknown params

$G_{\pi \times k}$ Matrix of 1st derivatives.

The explained sum of squares. (ESS) in this regression is

$$\hat{y}'\hat{y} \quad \text{or} \quad \cancel{\dot{d}_T' G(\theta)} \quad \hat{\dot{d}}_T' \hat{\dot{d}}_T$$

$$\hat{\dot{d}}_\pi = G (G^T G)^{-1} G^T \dot{d}_T$$

$$= \dot{d}_\pi^T G (G^T G)^{-1} G' G (G'G)^{-1} G' \dot{d}_T$$

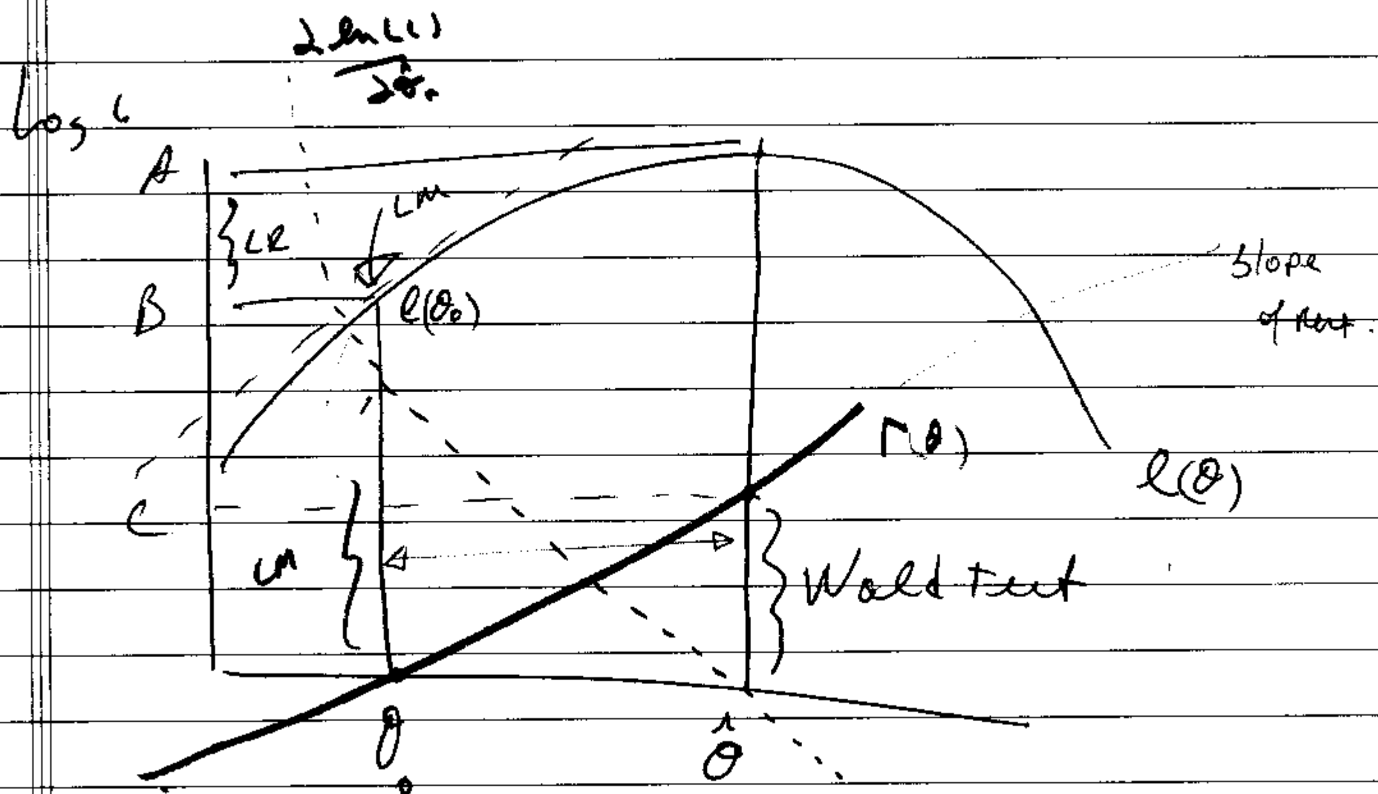
$$= \underbrace{\dot{d}_T' G'}_{} (G'G)^{-1} G' \dot{d}_T$$

$$= g' (G'G)^{-1} g'$$

Recall $TSS = ESS + SSE$

$TSS = \pi$ in this case $\dot{d}_\pi^T \dot{d}_\pi = \pi$

$\therefore ESS = \pi - SSE$!



$LR - 2(A-B) \sim \chi^2_J$

Note: At $r(\theta_0)$ the function is zero
At $r(\hat{\theta})$ it is not

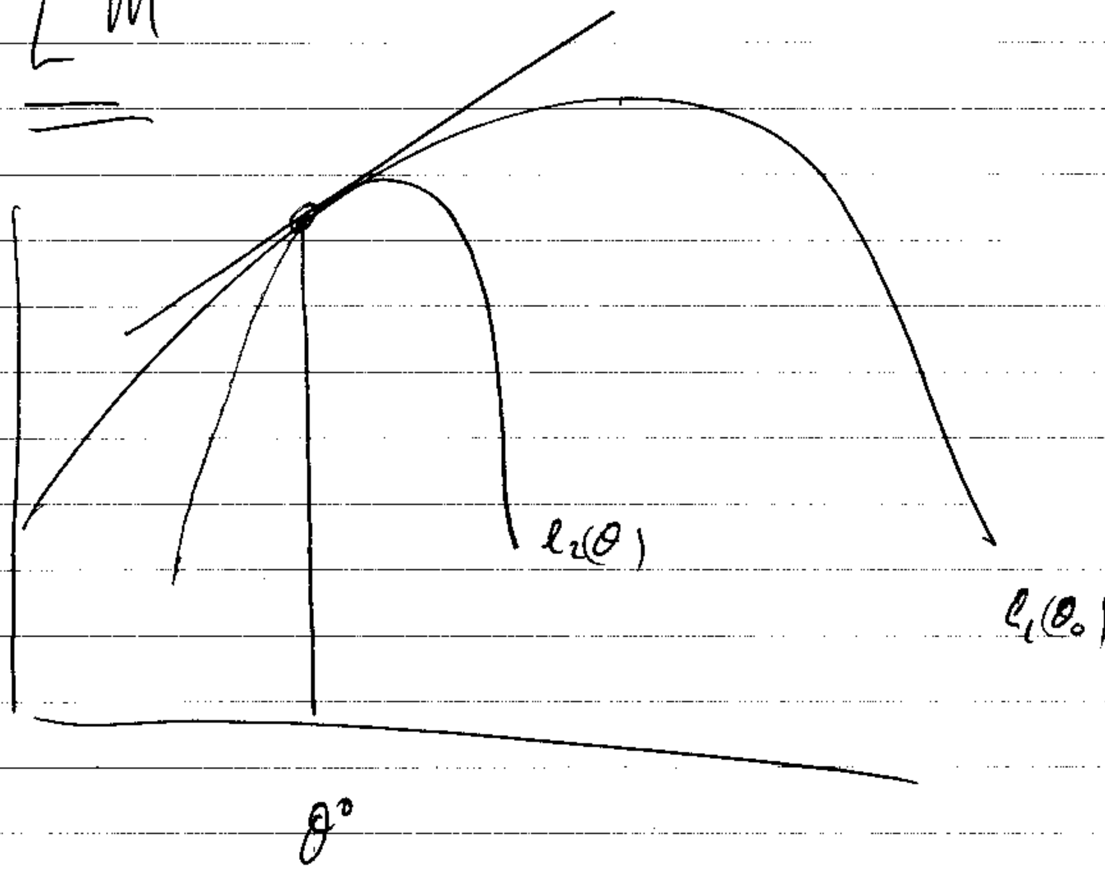
Wald $C - 0$

Relative to the functions curvature
at $r(\hat{\theta})$

LM - Looks at slope of tangent
at $\ell(\theta_0)$

Curvature of function is measured
at θ_0

LM



Closer to max with more curved function

When ℓ is flat $I(\theta_0)$ is small
$I(\theta_0)^{-1}$ is Big
and you are
more likely
to reject

$LM = g_0' I(\theta_0)^{-1} g_0$

slope (—)