# Ordered Probit

$$y_t^o = X_t \beta + u_t \qquad u_t \sim N(0,1)$$

$y_t^o$ is "latent", meaning its value is not directly observed.
Instead, we observe $y_t$ which takes on a limited number of values, e.g. 1, 2, 3.

These values are ordered, but their numerical value doesn't actually represent the exact numerical value of the latent variable.

Bond Ratings

Attitudes

For simplicity, let's assume 3 values for $y_t$

$$y_t = \begin{cases} 0 & \text{if } y_t^\circ < \gamma_1 \\ 1 & \text{if } \gamma_1 \le y_t^\circ < \gamma_2 \\ 2 & \text{if } y_t^\circ \ge \gamma_2 \end{cases}$$

$\gamma_1, \gamma_2$ are Threshold parameters.

If $y_t^\circ$ is below $\gamma_1$ we observe level 0. If its greater Than $\gamma_2$ we observe $y_t = 2$. In between, $y_t = 1$

$\underline{X_t}$ is $1 \times k$ (~~includes the constant~~)

If $X_t$ includes a constant, Then $\gamma_1$ & $\gamma_2$ will not both be identified. This leads to several parameterization of this model (all substantively equivalent.)

Usual solution is to exclude constant ($\beta_1 = 0$)

The log likelihood consists of adding the log probabilities for each case.

$$\Pr(y_t = 0) = \Pr(y_t^\circ < \gamma_1) = \Pr(X_t\beta + u_t < \gamma_1)$$
$$= \Pr(u_t < \gamma_1 - X_t\beta) = \underset{\uparrow\text{normal CDF}}{\Phi}(\gamma_1 - X_t\beta)$$

$$\Pr(y_t = 2) = \Pr(y_t^\circ \geq \gamma_2) = \Pr(X_t\beta + u_t \geq \gamma_2)$$
$$= \Pr(u_t \geq \gamma_2 - X_t\beta)$$
$$= 1 - \Phi(\gamma_2 - X_t\beta)$$
$$= \Phi(X_t\beta - \gamma_2) \quad \text{(by symmetry)}$$

$$\Pr(y_t = 1) = 1 - \Pr(y_t = 0) - \Pr(y_t = 2)$$
$$= 1 - \Phi(\gamma_1 - X_t\beta) - \Phi(X_t\beta - \gamma_2)$$
$$= \Phi(\gamma_2 - X_t\beta) - \Phi(\gamma_1 - X_t\beta)$$

$$\ell(\beta, \gamma_1, \gamma_2) = \sum_{y_t = 0} \log\left(\Phi(\gamma_1 - X_t\beta)\right) + \sum_{y_t = 2} \log\left(\Phi(X_t\beta - \gamma_2)\right)$$
$$+ \sum_{y_t = 1} \left(\log\left(\Phi(\gamma_2 - X_t\beta) - \Phi(\gamma_1 - X_t\beta)\right)\right)$$

$$\frac{\partial P_t}{\partial x_i}\bigg|_{y_t=0} = -\phi(\hat{r}_1 - X_t\hat{\beta})\hat{\beta}_i$$

$\phi$ is normal pdf.

$$\frac{\partial P_t}{\partial x_i}\bigg|_{y_t=2} = \phi(\hat{r}_2 - X_t\hat{\beta})\beta_i$$

$$\frac{\partial P_t}{\partial x_i}\bigg|_{y_t=1} = \left[\phi(\hat{r}_1 - X_t\hat{\beta}) - \phi(\hat{r}_2 - X_t\hat{\beta})\right]\beta_i$$

mfx, at(outcome=2)