$$\ln(\text{wage})_{it} = \beta_{1i} + \beta_2 S_{it} + \beta_3 \text{exper}_{it} + u_{it}$$

β_2

Random Effects

~~β_{1i}~~ β_{1i} treated as random.

in this model these time invariant differences must be exogenous w/ respect to S_{it}, exper_{it}.

$$y_{it} = \underset{k-1 \times 1}{x_{it}^T \beta_2} + (\alpha + \alpha_i) + u_{it}$$

x_{it}^T k-1 explanatory vars

β_2 k-1 parameters

α constant

α_i : Random effect for i.

$\alpha_i = z_i^T \underset{\sim}{\alpha} - E[z_i^T \underline{\alpha}]$ — Random. for individual.

$E[u_{it} \mid x_{it}] = 0$

$V[u_{it} \mid x_{it}] = \sigma_u^2$

$Cov(u_{it} \mid \alpha_i) = 0$

$Cov(\alpha_i, \alpha_j) = 0 \quad i \neq j$

$\boxed{E[\alpha_i \mid x_{it}] = 0}$

$V(\alpha_i \mid x_{it}) = \sigma_\alpha^2$

$Cov(u_{it}, u_{js}) = 0 \quad \forall i, j, t, s \quad i \neq j \quad t \neq s.$

$\eta_{it} = u_{it} + \alpha_i$

$V(\eta_{it}) = Var(u_{it}) + Var(\alpha_i)$

$= \sigma_u^2 + \sigma_\alpha^2$

$\cancel{\forall t \neq} \quad Cov(\eta_{it}, \eta_{is}) = \sigma_\alpha^2$

$Cov(\eta_{it}, \eta_{js}) = 0 \quad i \neq j \quad \forall t, s.$

$$y = X\beta + \underbrace{\alpha \otimes i_T + u}_{\text{error}} \quad \eta$$

$$\alpha^T = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n)^T$$

$$X^T = (X_1^T \ X_2^T \ \cdots \ X_n^T)^T$$

$$\mathrm{Cov}(\eta) = E\left[\{(\alpha \otimes i_T) + u\}\{(\alpha \otimes i_T) + u\}^T\right]$$

$$= \sigma_\alpha^2 I_n \otimes i_T i_T^T + \sigma_u^2 I_{nT}$$

$$\begin{bmatrix} \sigma_\alpha^2 & \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 & 0 & 0 & 0 & \cdot \\ \vdots & & & & & & & \\ \sigma_\alpha^2 & \cdot & & \sigma_\alpha^2 & 0 & 0 & - & \\ 0 & & & 0 & \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 & \\ 0 & & & 0 & \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 & \\ 0 & & & 0 & 0 & & 0 & \\ 0 & & & 0 & 0 & & 0 & \\ 0 & & & & 0 & & & \end{bmatrix} + \begin{bmatrix} \sigma_u^2 & & \\ & \ddots & \\ & & \sigma_u^2 \end{bmatrix}$$

For the i^{th} individual.

$$\operatorname{Cov}(\underset{\sim}{\eta}_i) = \begin{bmatrix} \sigma_\alpha^2+\sigma_u^2 & & & \sigma_\alpha^2 \\ \cancel{\alpha}\,\sigma_\alpha^2 & \sigma_u^2+\sigma_\alpha^2 & & \\ \sigma_\alpha^2 & & \ddots & \\ \sigma_\alpha^2 & & \sigma_\alpha^2 & \sigma_u^2+\sigma_\alpha^2 \end{bmatrix}$$

same for each person or individual.

Complete Model

$$\begin{pmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \\ \vdots \\ \underset{\sim}{y}_n \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \underset{\sim}{\beta} + \begin{pmatrix} \alpha_1 \underset{\sim}{i}_T \\ \alpha_2 \underset{\sim}{i}_T \\ \vdots \\ \alpha_n \underset{\sim}{i}_T \end{pmatrix} + \begin{pmatrix} \underset{\sim}{u}_1 \\ \underset{\sim}{u}_2 \\ \vdots \\ \underset{\sim}{u}_n \end{pmatrix}$$

X_i now include a constant. $X_{i\; T\times k}$ $\underset{\sim}{\beta}_{\,k\times 1}$

$$= \sigma_\alpha^2 I_n \otimes \underset{\sim}{i}_T \underset{\sim}{i}_T^T + \sigma_\mu^2 I_{nT}$$

$$= \sigma_\alpha^2 I_n \otimes \underset{\sim}{i}_T \underset{\sim}{i}_T^T + \left(\sigma_\mu^2 I_n \otimes I_T \right)$$

$$= I_n \otimes \underbrace{\left[\sigma_\alpha^2 \underset{\sim}{i}_T \underset{\sim}{i}_T^T + \sigma_\mu^2 I_T \right]}_{V}$$

$$\Phi = I_n \otimes V$$

$$V = \begin{bmatrix} \sigma_\alpha^2 + \sigma_\mu^2 & \sigma_\alpha^2 & & \sigma_\alpha^2 \\ \sigma_\alpha^2 & \sigma_\alpha^2 + \sigma_\mu^2 & & \\ & & \ddots & \sigma_\alpha^2 \\ \sigma_\alpha^2 & & & \sigma_\alpha^2 + \sigma_\mu^2 \end{bmatrix}$$

$$\hat{\beta} = (X^T \underset{nT \times nT}{\Phi^{-1}} X)^{-1} X^T \Phi^{-1} y \qquad \text{GLS estimate.}$$

required since Φ is NOT A diagonal.

$$\Rightarrow \Phi^{-1} = (I_n \otimes V)^{-1} = I_n \otimes \underset{T \times T}{V^{-1}}$$

It can be shown:

$$V^{-1} = \frac{i_T i_T^T}{T\sigma_v^2} + \frac{D_T}{\sigma_\mu^2}$$

$$\sigma_v^2 = T\sigma_\alpha^2 + \sigma_\mu^2$$

$$D_T = I_T - i_T i_T^T / T$$

Contains unknowns – GLS is NOT Feasible. –

Becomes Feasible By estimating σ_μ^2 & σ_v^2.

Find A Transformation, P s.t.

$$P \Phi P^T = \sigma_\mu^2 I_{nT}$$

$$P = I_n \otimes P^*$$

$$P^* = I_T - \gamma\, i_T i_T^T / T \qquad \gamma = 1 - \sigma_\mu / \sigma_v$$

$$\sigma_v = \sqrt{\sigma_\mu^2 + T\sigma_\alpha^2}$$

"quasi-demean"

Tests:

(1) are α_i Random?

If NOT, then Pooled least squares is efficient.

If Are, then Random effects is Probably more eff than Pooled LS. (with clusters).

(2) Test for possible correlation Between α_i and the Regressors.

If correlated – MUST use FE

Pooled, Between, and RE are all inconsistent in this case.

$$P^* \underset{\sim}{y}_i = \left(I_T - \gamma\, i_T i_T^T / T \right) \underset{\sim}{y}_i$$

$$= \underset{\sim}{y}_i - \gamma\, \underset{\sim}{i}_T\, i_T^T \underset{\sim}{y}_i / T$$

$$= \underset{\sim}{y}_i - \gamma\, \underset{\sim}{i}_T\, \bar{y}_{i\cdot}$$

So, $\sigma_\alpha^2 = 0 \Rightarrow$ Pooled least squares.

$\gamma = 0$

<u>Estimate γ</u>

(i) σ_u^2

(ii) σ_α^2 or σ_v^2

Fixed Effects: $\hat{\sigma}_u^2 = \dfrac{\hat{u}^T \hat{u}}{nT - k - n - 1}$ $\quad \underset{\sim}{\hat{u}} = \left(M_D \underset{\sim}{y} - M_D X \hat{\beta}_{FE} \right)$

Fixed effects or within residuals.

$\hat{\sigma}_v^2$: Comes from "Between Estimator"

$$i=1,\dots,n \qquad \bar{y}_{i\cdot} = \alpha + \bar{x}_{i\cdot}^T \beta + \underbrace{\alpha_i + \bar{u}_{i\cdot}}_{v_{i\cdot}}$$

i^{th} individual.

$$\bar{y}_{i\cdot} = \sum_{t=1}^{T} y_{it} / T$$

$$\mathrm{Var}(v_{i\cdot}) = \mathrm{Var}(\bar{u}_{i\cdot} + \alpha_i) = \sigma_u^2 / T + \sigma_\alpha^2$$

Take the least squares residuals from Between estimation

$$\hat{\sigma}_v^2 / T = \hat{\sigma}_u^2 / T + \hat{\sigma}_\alpha^2$$

$$\hat{\sigma}_v^2 = \hat{\sigma}_u^2 + T \hat{\sigma}_\alpha^2$$

Estimate $\hat{\sigma}_\alpha^2$ this way it may not be > 0

If that happens - model misspec.

And need to reconsider the model.