Model: All variation is captured in the Error Term.

$$y_{it} = \sum_{k=1}^{K} \beta_k x_{kit} + e_{it} \qquad \begin{array}{l} i = 1, \cdots N \\ t = 1, \cdots T \end{array}$$

note: β is the same for each individual in every time period. If any differences occur, they must be captured in the error term.

Individual i

$$\underset{\sim}{y_i} = X_i \beta + e_i \qquad i = 1, \cdots N$$

$$y_i = (y_{i1}, y_{i2}, \cdots y_{iT})'$$

$$X_i = \begin{bmatrix} 1 & x_{2i1} & x_{3i1} & \cdots & x_{ki1} \\ 1 & x_{2i2} & \ddots & & \vdots \\ \vdots & \vdots & & & \vdots \\ 1 & x_{2iT} & x_{3iT} & \cdots & x_{kiT} \end{bmatrix} \qquad \underset{\sim}{e_i} = \begin{bmatrix} e_{i1} \\ \vdots \\ e_{iT} \end{bmatrix}$$

$$\beta = [\beta_1, \beta_2 \cdots \beta_K]'$$

System

$$\begin{bmatrix} \underset{\sim}{y_1} \\ \underset{\sim}{y_2} \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} [\underset{\sim}{\beta}] + \underset{\sim}{e} \Rightarrow \underset{TN\times 1}{\underset{\sim}{y}} = \underset{TN\times K}{X} \underset{K\times 1}{\underset{\sim}{\beta}} + \underset{TN\times 1}{\underset{\sim}{e}}$$

$$E(ee') = \underset{NT\times NT}{\Omega} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1N} \\ \Omega_{21} & \Omega_{22} & \ddots & \vdots \\ \vdots & & & \vdots \\ \Omega_{N1} & & \cdots & \Omega_{NN} \end{bmatrix} \qquad \begin{array}{l} \underset{T\times T}{\Omega_{ii}} = E(\underset{\sim}{e_i} \underset{\sim}{e_i'}) \\ \\ \underset{T\times T}{\Omega_{ij}} = E(e_i e_j') \end{array}$$

Errors can be related in 3 ways

(1) Error Variance is different for each unit (Hetero)

(2) Errors are contemporaneously related across units

(3) Errors are related to one another across time.

These lead to the following Assumptions

(1) $E(e_{it} e_{it}') = \sigma_i^2 = \sigma_{ii} \quad \forall t$

(2) $E(e_{it} e_{jt}') = \sigma_{ij} \qquad i \neq j$

or $E(e_{is} e_{it}') = \sigma_{ij}$ if $i \neq j$
0 if $t \neq s$

(3) $e_{it} = \rho e_{it-1} + u_{it}$

$E(u_{it}) = 0$

$E[u_{is} u_{jt}'] = \begin{cases} \phi_{ij} & s = t \\ 0 & s \neq t \end{cases}$

Now, Figure out what Ω looks like, estimate it, AND Then do FGLS.

(1) State e_{it} in (3) above in terms of u_{it}

$e_{it} = \rho e_{it-1} + u_{it} \qquad e_{it-1} = \rho e_{it-2} + u_{it-1}$

$e_{it} = u_{it} + \rho u_{it-1} + \rho^2 e_{it-2}$

$e_{it} = \sum_{j=0} \rho^j u_{it-j} + \rho^{j+1} e_{it-j-1}$

$e_{it} = \sum_{j=0}^{\infty} \rho^j u_{i,t-j} \qquad$ if $|\rho| < 1$

(2) calculate (2) Based on result above

$$E(e_{it}\, e_{jt}) = E\left(\sum_{s=0}^{\infty} \rho_i^s u_{i,t-s} \cdot \sum_{w=0}^{\infty} \rho_j^w u_{j,t-w}\right)$$

$$\sigma_{ij} = E\left[\sum_{\substack{s=0\\ s=w}}^{\infty} \rho_i^s \rho_j^s\, u_{i,t-s})(u_{j,t-w})\right] + \underbrace{c.p.t}_{=0}$$

By (3) ABOVE.

$$= \phi_{ij} \sum_{s=0}^{\infty} \rho_i^s \rho_j^s$$

$$= \phi_{ij} \sum_{s=0}^{\infty} (\rho_i \rho_j)^s \qquad \text{if } |\rho_i \rho_j| < 1 \text{ Then}$$

$$\sigma_{ij} = \phi_{ij} \cdot \frac{1}{1-\rho_i\rho_j}$$

$$E\, e_{it} \cdot e_{j,t+s} = E\left[\sum_{s=0}^{\infty} \rho_i^s u_{i,t-s} \sum_{w=0}^{\infty} \rho_j^w u_{j,t+s-w}\right]$$

$$= \rho_j^s \frac{\phi_{ij}}{1-\rho_i\rho_j}$$

since $e_{i,t+s}$ is related to $e_{j,t}$. By factor of ρ_j^s

$$\Omega_{ij} = \frac{\phi_{ij}}{1-\rho_i\rho_j}\begin{bmatrix} 1 & \rho_j & & \rho_j^{T-1} \\ \rho_i & 1 & & \\ \rho_i^2 & & 1 & \vdots \\ \rho_i^{T-1} & \rho_i^{T-2} & \cdots & 1 \end{bmatrix}$$

note: Ω_{ij} is NOT symmetric.

Estimation

1. First, eliminate AR(1), Try using P-W Transformation.

$$P_i' = \begin{bmatrix} \sqrt{1-\rho_i^2} & & & \\ -\rho_i & 1 & & \\ & -\rho_i & 1 & \\ & & \ddots & \\ & & -\rho_i & 1 \end{bmatrix} \qquad P_i' y = P_i' X\beta + P_i' e$$

$$P_i e_i = \begin{bmatrix} \sqrt{1-\rho_i^2}\, e_i \\ u_{i2} \\ u_{i3} \\ \vdots \\ u_{iT} \end{bmatrix}$$

$$E\,[P_i e_i] = \begin{bmatrix} E\sqrt{1-\rho^2}\,e_i \\ 0 \quad u \\ \vdots \\ 0 \end{bmatrix} = \left[\begin{matrix} 0 \\ \sim \end{matrix}\right]$$

Proof:

$$\mathrm{Var}\ e^* = E\left(\sqrt{1-\rho^2}\, e_i\, e_i' \sqrt{1-\rho^2}\right) = E\,(1-\rho^2)\, e_1^2 = 1-\rho^2 \cdot \sigma_e^2$$

$$= 1-\rho^2 \cdot \frac{\sigma_v^2}{1-\rho^2} = \sigma_v^2$$

$$E\,(e_1 e_{1+j}) = 0$$

$$E\ (\rho e_0 + u_1)(\rho e_1 + u_1) = 0$$

$$E\left[\sqrt{1-\rho^2}\, e_1\, u_t\right] = 0$$

where $e_1 = \rho e_0 + u_1$ since $E\, u_i u_j = 0 \quad i \neq j$

Intuitively, e_1 happens before $u_2, \cdots$. Therefore u_2 can have no effect on prior events.

Transform each equation:

$$E\,(P_i e_i)(P_j e_j)' = \begin{bmatrix} ? & \Phi_{ij} & & \\ & & \Phi_{ij} & \\ & & & \ddots \end{bmatrix}_{T-1 \times T-1}$$

cov. between transform errors of ith and jth eq.

The first element $E\left[\sqrt{1-\rho_i^2}\, e_{i1} \sqrt{1-\rho_j^2}\, e_{j1}\right] =$

$$\sqrt{1-\rho_i^2}\,\sqrt{1-\rho_j^2}\ \frac{\Phi_{ij}}{1-\rho_i\rho_j} \neq \Phi_{ij} \quad \text{unless } \rho_i = \rho_j$$

Eliminate 1st row of P. matrix – Transform system based on T-1 eq for each individual.

$E\, e_i^* = 0$

$E\, e_i^{*\prime} e_j^* = \Phi_{ij}\, I_{T-1}$

$\mathrm{Cov}\; e_i^* = \Phi_{ii}\, I_{T-1}$

Only the contemporaneous correlation remains

$$\begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_N^* \end{bmatrix} = \begin{bmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_N \end{bmatrix} \beta + \begin{bmatrix} e_1^* \\ e_2^* \\ \vdots \\ e_N^* \end{bmatrix}$$

Let $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1N} \\ \Phi_{21} & & & \\ \vdots & & & \\ \Phi_{N1} & & & \Phi_{NN} \end{bmatrix}$

$\mathrm{cov}\; e^* = \Phi \otimes I_{T-1}$

$$\tilde\beta_{GLS} = \left(X_*'(\Phi \otimes I_{T-1})X^*\right)^{-1} X_*(\Phi \otimes I_{T-1})^{-1} y^*$$

$$\tilde\beta_{FGLS} = \left(X_*'(\hat\Phi \otimes I_{T-1})^{-1} X_*\right)^{-1} X_*(\hat\Phi \otimes I_{T-1})^{-1} y^*$$

$\hat\Phi = \begin{bmatrix} \hat\Phi_{11} & & \\ \hat\Phi_{21} & \ddots & \\ \vdots & & \\ \hat\Phi_{N1} & \cdots & \hat\Phi_{NN} \end{bmatrix}$ $\qquad \hat\Phi_{ij} = \dfrac{\hat{\hat e}_{\cdot i}\, \hat{\hat e}_{\cdot j}}{T-1}$

1. OLS - calculate $\hat e$ and ρ_i for each i
2. Form ρ eliminate row 1 for each N calculate $\hat{\hat e}$
3. use $\hat{\hat e}$ to estimate contemp. cov. Apply SUR.

Models where intercept varies across individuals
(But constant for each individual over time)

$$(1) \quad y_{it} = \beta_{1i} + \sum_{k=2}^{K} \beta_k X_{kit} + e_{it}$$

$$E(e_{it}) = 0 \qquad i = 1, \cdots n$$
$$E(e_{it}^2) = \sigma_e^2 \qquad t = 1, \cdots T.$$

This model requires that we stack by individual. Each individual has his own intercept.

$$\underset{NT \times 1}{\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ y_N \end{bmatrix}} = \underset{NT \times N+K-1}{\begin{bmatrix} j_T & 0 & \cdots & 0 & 0 & X_1 \\ 0 & j_T & & & & X_2 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & & j_T & & \\ 0 & \cdots & & & j_T & X_N \end{bmatrix}} \underset{N+K-1 \times 1}{\begin{bmatrix} \beta_{11} \\ \beta_{12} \\ \vdots \\ \beta_{1N} \\ \tilde{\beta}_s \end{bmatrix}} + \underset{NT \times 1}{\begin{bmatrix} \tilde{e}_1 \\ \vdots \\ e_n \end{bmatrix}}$$

X_i = observations of X for i^{th} individual.
$i = 1, \cdots N.$

$$\tilde{y} = \left[I_N \otimes j_T \;\vdots\; X \right] \begin{bmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_s \end{bmatrix} + \tilde{e}$$

Let $\bar{X} = [I_N \otimes j_T \,\vdots\, X]$

$$\tilde{b} = (\bar{X}'\bar{X})^{-1}\bar{X}'y$$

$$\tilde{b} = \left[(I_N \otimes j_T \vdots X)' (I_N \otimes j_T \vdots X) \right]^{-1} (I_N \otimes j_T \vdots X)' y$$

$$\tilde{b} = \left(\begin{bmatrix} (I_N \otimes j_T)' \\ (X)' \end{bmatrix} (I_N \otimes j_T \vdots X) \right)^{-1} \begin{bmatrix} (I_N \otimes j_T)' \\ (X)' \end{bmatrix} y.$$

$$(I_N \otimes j_T) = (I_N' \otimes j_T')$$

$$\underset{\sim}{b} = \begin{bmatrix} I_N \otimes j_T' j_T & (I_N \otimes j_T)' X \\ X'(I_N \otimes j_T) & X'X \end{bmatrix}^{-1} \bar{X}' y$$

This inversion can be computationally impossible.
We can use the partitioned inverse result to simplify matters.

$$(X'X)^{-1} = \begin{bmatrix} P & R \\ R' & Q \end{bmatrix}^{-1} = \begin{bmatrix} P^{-1} + P^{-1} R F R' P^{-1} & -P^{-1} R F \\ -F R' P^{-1} & F \end{bmatrix}$$

where $F \equiv \left[Q - R' P^{-1} R \right]^{-1}$

$$\underset{\sim}{b} = (X'X)^{-1} X' y = \begin{bmatrix} \underset{\sim}{b_1} \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{T} I_N + \frac{1}{T}(I_N \otimes j_T)' X F X' (I_N \otimes j_T)\frac{1}{T} & -\frac{1}{T}(I_N \otimes j_T)' X F \\ -F X'(I_N \otimes j_T)\frac{1}{T} & F \end{bmatrix}$$

$$F^{-1} = X'X - X'(I_N \otimes j_T)\frac{1}{T}(I_N \otimes j_T)' X$$

$$X' \tfrac{1}{T}\left(I_{NT} - (I_N \otimes j_T j_T')\right) X$$

$$I_N \otimes I_T = I_{NT}$$

$$X' \left(I_N \otimes I_T - I_N \otimes \frac{j_T j_T'}{T} \right) X$$

$$F^{-1} = X' \left(I_N \otimes I_T - \frac{j_T j_T'}{T} \right) X = X'(I_N \otimes D_T) X$$

where $D_T = I_T - \frac{j_T j_T'}{T}$

$$\cancel{\left[-F X'(I_N \otimes j_T)\tfrac{1}{T} = -X'(I_N \otimes D_T) X X'(I_N \otimes j_T) \right]}$$

$$b_s = -F'R'P^{-1}Xy + FXy \qquad F = F'$$

$$= -FX'(I_N \otimes j_T)\tfrac{1}{T}(I_N \otimes j_T)y + FX'y$$

$$= -FX'\left(I_N \otimes \frac{j_T j_T'}{T}\right)y + FX'y$$

$$= FX'\left(I_{NT} - I_N \otimes \frac{j_T j_T'}{T}\right)y = FX'\left(I_N \otimes I_T - I_N \otimes \frac{j_T j_T'}{T}\right)y$$

$$= FX'\left(I_N \otimes I_T - \frac{j_T j_T'}{T}\right)y$$

$$\underset{\sim}{b_s} = FX'(I_N \otimes D_T)y$$

$$\boxed{b_s = \left[X'(I_N \otimes D_T)X\right]^{-1} X'(I_N \otimes D_T)y} \quad (2)$$

** note that this requires inversion of an $(K-1 \times K-1)$ MATRIX.

Intercept

$$\underset{\sim}{b_1} = \left[\tfrac{1}{T}I_N + \tfrac{1}{T}(I_N \otimes j_T)'XFX'(I_N \otimes j_T)\tfrac{1}{T}\right](I_N \otimes j_T)'y$$

$$- \tfrac{1}{T}(I_N \otimes j_T)'XFX'y$$

$$\underset{\sim}{b_1} = \tfrac{1}{T}(I_N \otimes j_T)'\underset{\sim}{y} + \tfrac{1}{T}(I_N \otimes j_T)'XFX'\left(I_N \otimes \frac{j_T j_T'}{T}\right)\underset{\sim}{y}$$

$$- \tfrac{1}{T}(I_N \otimes j_T)'XFX'\underset{\sim}{y}$$

First Term

$$\tfrac{1}{T}(I_N \otimes j_T)'y = \frac{1}{T}\begin{bmatrix} j_T' & & \\ & j_T' & \\ & & \ddots \\ & & & j_T' \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \frac{j_T' y_1}{T} \\ \frac{j_T' y_2}{T} \\ \vdots \\ \frac{j_T' y_N}{T} \end{bmatrix} = \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \\ \vdots \\ \bar{y}_{N\cdot} \end{bmatrix} = \bar{y}_{\circ}$$

$\bar{y}_{i\cdot}$ is average over the i^{th} individual.

Likewise, in the next 2 terms

$$\frac{1}{T}(I_N \otimes j_T)'X =$$

$$\frac{1}{T}\begin{bmatrix} j_T' & & \\ & j_T' & \\ & & j_T' \end{bmatrix}_{N \times NT} X = \begin{bmatrix} \bar{X}_{1\cdot}' \\ \bar{X}_{2\cdot}' \\ \vdots \\ \bar{X}_{N\cdot}' \end{bmatrix}_{(n \times k-1)} = \bar{X}_{\cdot}'$$

$$b_1 = \bar{y}_{\cdot} + \bar{X}_{\cdot}' F X \left(I_N \otimes \frac{j_T j_T'}{T}\right) y$$

$$- \bar{X}_{\cdot}' F X' y$$

$$= \bar{y}_{\cdot} - \bar{X}_{\cdot}' \left[-FX\left(I_N \otimes \frac{j_T j_T'}{T}\right) y + FX'y \right]$$

$$\underset{\sim}{b}_1 = \bar{y}_{\cdot} - \bar{X}_{\cdot}' \underset{\sim}{b}_s$$

Properties of D_T

(i) D_T is symmetric $\quad D_T = D_T'$

(ii) D_T is Idempotent $\quad D_T \cdot D_T = D_T$

If D_T is (i) and (ii) then so is $(I_N \otimes D_T)$. $\square$

$$b_s = \left[X'(I_N \otimes D_T)(I_N \otimes D_T)X\right]^{-1} X'(I_N \otimes D_T)(I_N \otimes D_T) y$$

$(W'W)^{-1}W z$ or like in the book

$(Z'Z)^{-1}Z'W$

$Z \equiv (I_N \otimes D_T)X$

$W \equiv (I_N \otimes D_T)y$

Multiplication of any vector by D_T is equivalent to putting it in deviation form.

$$D_T \underset{\sim}{y}_i = \left[I_T - \frac{j_T j_T'}{T} \right] y_i = \underset{\sim}{y}_i - \frac{j_T j_T'}{T} \underset{\sim}{y}_i$$

$$= \underset{\sim}{y}_i - \underset{\sim}{j}_T \bar{y}_i$$

Transform the model using $(I_N \otimes D_T)$

$$(I_N \otimes D_T) y = (I_N \otimes D_T) X \beta + (I_N \otimes D_T) e$$

$$W = [I_N \otimes D_T] \left[I_N \otimes j_T \;\vdots\; X_s \right] \beta + [\quad] e$$

$$= I_N \otimes D_T j_T \quad (I_N \otimes D_T) X_s$$

$$\left(I_T - \frac{j_T j_T'}{T} \right) j_T = j_T - j_T \frac{T}{T} = 0 \quad \therefore$$

$$W = Z\beta + e^* \qquad \text{where } e^* = (I_N \otimes D_T) e$$

Within Estimator Becomes

$$\hat{\beta} = (Z'Z)^{-1} Z'W$$

Everything else is obtained as always

$$\hat{e} = y - X \begin{bmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_2 \end{bmatrix} \qquad \hat{\sigma}_e^2 = \frac{\hat{e}'\hat{e}}{NT - N + K - 1}$$

Model: Constant Slopes, Individual differences where individuals represent a random sample.

Error Components

$$y_{it} = \beta_{1i} + \sum_{k=2}^{K} \beta_k x_{ikt} + e_{it} \qquad \begin{array}{l} i = 1, \cdots, n \\ t = 1, \cdots T. \end{array}$$

β_{1i} is now random

$E(\beta_{1i}) = \bar{\beta}_1$

$Var(\beta_{1i}) = \sigma_u^2$

$$\beta_{1i} = \bar{\beta}_1 + u_i$$

$E u_i = 0$
$V(u_i) = \sigma_u^2$
$E u_i u_j = 0$
$E u_{it}, u_{is} = 0 \quad t \neq s$

ith Individual

$$y_i = X_i \underset{\sim}{\beta} + (u_i j_T + e_i)$$

$$E\,(u_i j_T + e_i) = 0$$

$$E\,(u_i j_T + e_i)(u_i j_T + e_i)' =$$

$$= E\left(u_i j_T j_T' u_i' + u_i j_T e_i' + e_i j_T' u_i' + e_i e_i'\right)$$

$$\sigma_u^2 j_T j_T' + 0 + 0 + \sigma_e^2 I_T$$

Implications
1. homoskedastic with $\sigma_u^2 + \sigma_e^2$ on diagonal elements
2. "serial correlation" σ_u^2 is same for each time (for i).

Complete Model

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \beta + \begin{matrix} u_1 j_T \\ \vdots \\ u_n j_T \end{matrix} + \begin{matrix} e_1 \\ \vdots \\ e_n \end{matrix}$$

$$y = X\beta + u \otimes j_T + e$$

$$\text{Cov Matrix} = E(u \otimes j_T + e)(u \otimes j_T + e)'$$

$$= \sigma_u^2 I_N \otimes j_T j_T' + \sigma_e^2 I_{NT}$$

$$= \sigma_u^2 I_N \otimes j_T j_T' + \sigma_e^2 I_N \otimes I_T$$

$$= I_N \otimes \left[\sigma_u^2 J_T J_T' + \sigma_e^2 I_T\right]$$

$$\Phi = I_N \otimes V_{T\times T}$$

The idea now, is to find a matrix (nonsing) P that diagonalizes the covariance matrix

$$P \Phi P' = \sigma_e^2 I_{NT}$$

$$P = I_N \otimes P^* \qquad P^* = I_T - \left(1 - \frac{\sigma_e}{\sigma_1}\right)\frac{J_T J_T'}{T}$$

$$\sigma_1^2 = T\sigma_u^2 + \sigma_e^2 \qquad \alpha \in [0,1)$$

since if $\sigma_u^2 = 0$ – no indiv diff
then $\sigma_1^2 = \sigma_e^2 \Rightarrow P^* = I_T$

Hildreth Houck Random Coefficient Model

This type of model is reasonable when we have cross sectional data on a number of micro-units.

$$y_t = \sum_{k=1}^{K} \beta_{tk} X_{tk}$$

β varies with each individual.

$$\beta_{tk} = \bar{\beta}_k + v_{tk} \qquad k = 1, \ldots K.$$

$\bar{\beta}_k$: mean effect

v_{tk} : individual effect

What this amounts to is error components for slopes.

v_{tk} : $E(v_{tk}) = 0$

$Var(v_{tk}) = \alpha_k^2$

could make other covariance assumptions like presence of serial correlation + autocor. etc.

$$y_t = \sum_{k=1}^{K} (\bar{\beta}_k + v_{tk}) X_{tk}$$

$$= \sum_{k=1}^{K} \bar{\beta}_k X_{tk} + \sum_{k=1}^{K} v_{tk} X_{tk}$$

$$= X_t'\beta + X_t' v_t$$

since one column of X_t is $= 1$

$$\sum_{k=1}^{K} v_{tk} X_{tk} = v_{t1} + \sum_{k=2}^{K} v_{tk} X_{tk}$$

v_{t1} captures variability of intercept + overall variance in equation. Presence of e would be irrelevant.

$$E\, \chi_t' V_t = 0$$

$$E\, \chi_t' V_t V_t' \chi_t = \chi_t' \operatorname{Cov}(V_t) \chi_t$$

$$\operatorname{Cov} V_t = \begin{bmatrix} \alpha_1^2 & & & 0 \\ & \alpha_2^2 & & \\ & & \ddots & \\ 0 & & & \alpha_k^2 \end{bmatrix}$$

if $E\, V_t V_s \neq 0 \quad t \neq s$ then off diag elements are filled.

$$\operatorname{Var} \chi_t' V_t = \chi_t' \Sigma \chi_t = 1 \cdot \alpha_1^2 + \chi_{t2}^2 \alpha_2^2 + \cdots \chi_{tk}^2 \alpha_k^2$$

$$= \begin{bmatrix} 1 & \chi_{t2}^2 & \chi_{t3}^2 & \cdot & \chi_{tk}^2 \end{bmatrix} \begin{bmatrix} \alpha_1^2 \\ \alpha_2^2 \\ \vdots \\ \alpha_k^2 \end{bmatrix}$$

$$= z_t' \alpha$$

α can be estimated as

$$\hat{\alpha} = (F'F)^{-1} F' \dot{\hat{e}}$$

$$F = \dot{M} Z \qquad \dot{M} = \left(I - \overset{\circ}{X}(X'X)^{-1}X'\right)$$

$$\hat{e} = My = Me$$

$$\dot{\hat{e}} = \dot{M} y$$

Proof: $\hat{e} = Me \qquad \operatorname{Cov} \hat{e} = M \Phi M$

$$\operatorname{Cov}(\hat{e}) = \begin{bmatrix} \ddots & & 0 \\ & E\hat{e}_t^2 & \\ 0 & & \ddots \end{bmatrix} \qquad E\, \hat{e}_t^2 = M_{t\cdot}\, \Phi_t\, M_{t\cdot}$$

$$= M_{t\cdot}^2\, \Phi_t^2$$

$$E\,\dot{\hat{e}} = \dot{M} Z \alpha$$

$$\dot{\hat{e}} = \dot{M} Z \alpha + u$$

apply OLS

$$\left[(\dot{M} Z)'(\dot{M} Z)\right]^{-1} (\dot{M} Z)' \dot{\hat{e}} = \hat{\alpha}$$