Fixed Effects

Let $\underset{\sim}{y}_i$ and X_i be the T obs

for the i^{th} unit and let $\underset{\sim}{i}$ $T\times 1$

vector of 1's

$$\underset{\underset{T\times 1}{\sim}}{y_i} = \underset{T\times(k-1)}{X_i}\,\underset{(k-1)\times 1}{\beta} + \underset{\sim}{i}\,\alpha_i + \underset{\underset{T\times 1}{\sim}}{u_i}$$

α_i scalar

$\underset{\sim}{i}^T$ $T\times 1$ $= \{1\ 1\ 1\ \cdots 1\}^T$

Collecting all individuals

$$\begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \\ \vdots \\ \underset{\sim}{y}_N \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_N \end{bmatrix} \underset{\sim}{\beta} + \begin{bmatrix} \underset{\sim}{i} & 0 & 0 & \cdots & 0 \\ \underset{\sim}{0} & \underset{\sim}{i} & 0 & \cdots & 0 \\ & & & & \\ \underset{\sim}{0} & \cdots & & & \underset{\sim}{i} \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{bmatrix} \underset{\sim}{u}_1 \\ \vdots \\ \vdots \\ \underset{\sim}{u}_N \end{bmatrix}$$

$$\underset{\sim}{y} = \left[X \;\; \underset{\sim}{d}_1 \;\; \underset{\sim}{d}_2 \; \cdots \; \underset{\sim}{d}_N \right] \begin{matrix} \underset{\sim}{\beta} \\ \underset{\sim}{\alpha} \end{matrix} + \underset{\sim}{u}$$

$$\underset{\sim}{d}_{j\,NT\times 1} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

$$X_{NT\times \Sigma k_i} \qquad \underset{\sim}{\alpha}\;\; N\times 1 \qquad \underset{\sim}{u}\;\; NT\times 1$$

$$\underset{\sim}{\beta}\;\; (k-1)\times 1$$

$$\text{Let} \quad \underset{NT\times N}{D} \equiv \left[\underset{\sim}{d}_1 \;\; \underset{\sim}{d}_2 \; \cdots \; \underset{\sim}{d}_n \right]$$

$$\underset{\sim}{y} = X \underset{\sim}{\beta} + D \underset{\sim}{\alpha} + \underset{\sim}{u}$$ LSDV Model

use FWL to remove D.

$$M_D = I - D(D^T D)^{-1} D^T$$

$$M_D \underset{\sim}{y} = M_D X \beta + M_D D \underset{\sim}{\alpha} + M_D \underset{\sim}{u}$$

$$M_D y = M_D X \beta + M_D \underset{\sim}{u}$$ Within Model

$$\underset{\sim}{\hat\beta} = (X^T M_D X)^{-1} X^T M_D \underset{\sim}{y}$$

since $M_D = M_D^T$ and $M_D \cdot M_D = M_D$.

Notice, LS estimate of LSDV ($\tilde{\beta}$)
and $\tilde{\beta}$ in the within-Model
Are The <u>SAME</u>.

$$D^T D = \left[\begin{pmatrix} d_1 & d_2 & \cdots & d_n \end{pmatrix} \right]^T \begin{pmatrix} d_1 & d_2 & \cdots & d_N \end{pmatrix}$$

$$= \begin{pmatrix} d_1^T \\ d_2^T \\ \vdots \\ d_N^T \end{pmatrix} \begin{matrix} d_1 & d_2 & \cdots & d_N \end{matrix}$$

$$= \begin{bmatrix} d_1^T d_1 & d_1^T d_2 & \cdots & d_1^T d_N \\ d_2^T d_1 & d_2^T d_2 & \cdots & d_2^T d_N \\ \vdots & & & \\ d_N^T d_1 & \cdot & \cdot & d_N^T d_N \end{bmatrix}$$

$$= \begin{bmatrix} T & 0 & 0 & \cdot & 0 \\ 0 & T & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdot & \cdot & 0 & T \end{bmatrix}$$

$$(D^T D)^{-1} = \begin{bmatrix} \frac{1}{T} & & & 0 \\ & \frac{1}{T} & & \\ & & \ddots & \\ 0 & & & \frac{1}{T} \end{bmatrix}$$

$$D \cdot (D^T D)^{-1} = \begin{bmatrix} d_1/T & & & 0 \\ & d_2/T & & \\ & & \ddots & \\ 0 & & & d_N/T \end{bmatrix}$$

$$D \cdot (D^T D)^{-1} D^T = \begin{bmatrix} d_1 d_1^T/T & & & 0 \\ & d_2 d_2^T/T & & \\ & & \ddots & \\ 0 & & & d_N d_N^T/T \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{i} i^T/T & & & 0 \\ & \underset{\sim}{i} i^T/T & & \\ & & \ddots & \\ 0 & & & \underset{\sim}{i} i^T/T \end{bmatrix}$$

$$= I_N \otimes i i^T/T$$

Kronecker product $\otimes$

let $A = \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$ $\quad B = \begin{matrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{matrix}$

$$\underbrace{\underset{2\times 2}{A} \otimes \underset{2\times 3}{B}}_{4\times 6} = \underset{4\times 6}{\begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}}$$

$$= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{23} \\ a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\ \vdots & \vdots & \vdots & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} \\ & & & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \end{pmatrix}$$

Algebra

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

$$(A \otimes B)^{T} = A^{T} \otimes B^{T}$$

$$M_D = I_{NT} - \left(I_N \otimes \underset{\sim}{i}\underset{\sim}{i}^T / T \right)$$

$$= I_N \otimes D_T$$

where $D_T = I_T - \frac{1}{T} \underset{\sim}{i}\underset{\sim}{i}^T$

Homework: show this.

$$D_T \underset{\sim}{y}_1 = y_1 - \frac{1}{T} \underset{\sim}{i}\underset{\sim}{i}^T \underset{\sim}{y}_1$$

$$= \underset{\sim}{y}_1 - \underset{\sim}{i} \left(\underset{\sim}{i}^T \underset{\sim}{y}_1 / T \right)$$

$$= \underset{\sim}{y}_1 - \underset{\sim}{i} \, \bar{y}_1$$

D_T puts any vector in deviation from mean (taken over time) form.

So, the fixed effects regression looks like.

$$(y_{it} - \bar{y}_{i\cdot}) = (x_{it} - \bar{x}_{i\cdot})^T \underset{\sim}{\beta} + \text{res.}$$

The coefficients on the fixed effects can be recovered

$$\hat{\alpha} = (D^T D)^{-1} D^T (y - X\hat{\beta})$$

which implies that for each i

$$\hat{\alpha}_i = \bar{y}_{i.} - \bar{x}_{i.}^T \hat{\beta}$$

= time mean of y_i — time mean of x_i

The "dot" tells you over which dimension the average is taken.

Testing joint significance of α.

- LSDV vs pooled or Pop. average model.

H_0: $\alpha_1 = \alpha_2 = \alpha_3 \cdots = \alpha_N$

H_A: NOT H_0:

$$\lambda_2 = \frac{\left(SSE_p - SSE_{LSDV}\right) / N-1}{SSE_{LSDV} \cancel{\hat{\sigma}^2_{LSDV}} \,/\, NT-N-K-1}$$

$$\sim F_{N-1,\, NT-N-K-1} \text{ if } H_0 \text{ True.}$$