$$y = x\beta + u \qquad u \sim iid(0, \sigma^2 I)$$

Provided Rank $(x) = k \leq n$
LS is efficient among linear unbiased estimators of β.

$$\Rightarrow E[u | x] = 0$$

If instead $E[u_t | x_t] = 0$
then LS is consistent.

In many instances, u_t are correlated with x_t. (regressors for the same observation)

LS inconsistent

Suppose we can find a set of k or more variables s.t.

$$E[u_t | w_t] = 0$$

In this case, we can estimate β consistently using IV provided a couple of other things happen.

Correlation Between Errors & Regressors

(1) Errors-in-Variables

Measurement Error in y_t
Measurement Error in x_t

If Errors are in y_t, no Problems for LS. If Error is in x_t, then LS is inconsistent because this causes u_t to be correlated with observed Regressors.

Model

$$y_t^\circ = \beta_1 + \beta_2 x_t^\circ + u_t^\circ$$

$$u_t^\circ \sim iid(0, \sigma^2)$$

x_t° & y_t° not observed.

Instead

$$X_t = X_t^\circ + V_{1t}$$

$$y_t = y_t^\circ + V_{2t}$$

V_{1t} & V_{2t} are errors in measurement assoc with X_t, y_t

Assume these have variances ω_1^2, ω_2^2 s.t.

$$V_{1t} \sim iid(0, \omega_1^2)$$
$$V_{2t} \sim iid(0, \omega_2^2)$$

Substitution yields:

$$y_t - V_{2t} = \beta_1 + \beta_2(X_t - V_{1t}) + u_t^\circ$$

$$y_t = \beta_1 + \beta_2 X_t + (u_t + V_{2t} - \beta_2 V_{1t})$$

$$= \beta_1 + \beta_2 X_t + u_t$$

$$u_t = u_t^\circ + V_{2t} - \beta_2 V_{1t}$$

$$Var(u_t) = \sigma^2 + \omega_2^2 + \beta_2^2 \omega_1^2$$

Measurement error increases the Variance of y_t

Notice, the new error u_t is now correlated with x_t

$$Cov(u_t, x_t) = E[(u_t - E(u_t))(x_t - E(x_t))]$$

$$E(u_t) = 0$$
$$E(x_t) = x_t^\circ$$

So
$$E[(u_t^\circ + v_{2t} - \beta_2 v_{1t})(v_{1t})]$$
$$= E(-\beta_2 v_{1t}^2) = -\beta_2 \, Var \, v_{1t}^2 = -\beta_2 \omega_1^2$$

As long as x_t (x_t°) related to y_t, $Cov() \neq 0$ LS Inconsistent.

SEM

structural eq's

demand $q_t^d = \gamma_d p_t \; X_t^d \beta_d + u_t^d$

supply $q_t^s = \gamma_s p_t \; X_t^s \beta_s + u_t^s$

Equili $q_t^s = q_t^d$

X_t^d exog variables that determine demand

X_t^s exog vars that determine supply.

u_t^d demand shocks or errors

u_t^s supply shocks or errors

Rewrite, with endog on ~~RHS~~ LHS

$$q_t - \gamma_d p_t = X_t^d \underline{\beta}_d + u_t^d$$

$$q_t - \gamma_s p_t = X_t^s \underline{\beta}_s + u_t^s$$

1

$$\begin{bmatrix} 1 & -\gamma_d \\ 1 & -\gamma_s \end{bmatrix} \begin{matrix} q_t \\ p_t \end{matrix} = \begin{pmatrix} X_t^d & 0 \\ 0 & X_t^s \end{pmatrix} \begin{pmatrix} \beta_d \\ \beta_s \end{pmatrix} + \begin{matrix} u_t^d \\ u_t^s \end{matrix}$$

$$\begin{bmatrix} 1 & -\gamma_d \\ 1 & -\gamma_s \end{bmatrix} \begin{matrix} q_t \\ p_t \end{matrix} = \begin{matrix} X_t^d \beta_d + u_t^d \\ X_t^s \beta_s + u_t^s \end{matrix}$$

$$\begin{pmatrix} q_t \\ p_t \end{pmatrix} = \begin{bmatrix} 1 & -\gamma_d \\ 1 & -\gamma_s \end{bmatrix}^{-1} \begin{bmatrix} X_t^d \beta_d + u_t^d \\ X_t^s \beta_s + u_t^s \end{bmatrix}$$

Reduced Form Eq.

Notice, each endog variable is a function of all exog. vars and all errors in the structural model.

$\Rightarrow$ A shock to U_t^d or U_t^s affects P_t (a regressor in both structural Eq's.) Therefore errors are correlated with regressors of S & D!

Instruments

(1) Must be exogenous

$E[u|W] = 0$ strongly

or.

$E[u_t|W_t] = 0$ weakly.

(2) Must be relevant.

$$\underset{n \to \infty}{\text{plim}} \frac{1}{n} W^T X = S_{W^T X}$$

finite & Nonsingular.

$\Rightarrow$ Instruments Must be Correlated with X; ~~and~~ which includes the variables measured with error or that are endogenous.

Note: LLN $\Rightarrow$ $\underset{n \to \infty}{\text{plim}} \frac{W^T u}{n} = 0$

Simple IV estimator

Find W that is exogenous & relevant. If so, it should satisfy

$$E[W^T u] = 0 \qquad \text{wt. } \underset{=}{W}_{T \times K}$$

MoM $\underline{E}[W^T(\underset{\sim}{y} - X\underset{\sim}{\beta})] = 0$

set = to sample counterpart & solve.

$$W^T(y - X\hat{\beta}) = 0$$

$$W^T y - W^T X\hat{\beta} = 0$$

$$W^T X\hat{\beta} = W^T y$$

$$\hat{\beta}_{IV} = (W^T X)^{-1} W^T \underset{\sim}{y}$$

- generalized inverse since $W^T X$ not symmetric.

- Consistent as long as

$$\underset{n \to \infty}{\text{plim}} \frac{W^T u}{n} = 0 \quad (\text{By LLN})$$

Obviously, if w & x are not correlated, then the inverse $(w^T x)^{-1}$ ~~[illegible]~~ won't be successful (or is not highly correlated it will be very large).

$$\hat{\beta}_{IV} = \beta + (w^T x)^{-1} w^T \underset{\sim}{u}$$

~~plim~~

$$\hat{\beta}_{IV} = \underset{\sim}{\beta} + \left(\frac{w^T x}{n}\right)^{-1} \frac{w^T u}{n}$$

$$\underset{n \to \infty}{\text{plim}}\ \hat{\beta}_{IV} = \beta + \underset{n \to \infty}{\text{plim}} \left(\frac{w^T x}{n}\right)^{-1} \underset{n \to \infty}{\text{plim}} \frac{w^T u}{n}$$

$$\underset{\sim}{\beta} + S_{w^T x}^{-1} \cdot 0$$

$$\text{Cov}(\hat{\beta}_{IV}) = E\left[(\hat{\beta}_{IV} - E(\hat{\beta}_{IV}))(\hat{\beta}_{IV} - E(\hat{\beta}_{IV}))\right]$$

Asy Cov $(\hat{\beta}_{IV})$ — Replace with Plims

$$\underset{n \to \infty}{\text{plim}}\ E\left(\frac{w^T x}{n}\right)^{-1} \frac{w^T u}{n} \frac{u^T w}{n} \left(\frac{w^T x}{n}\right)^{-1^T}$$

$$\frac{\sigma^2 w^T w}{n^2}$$

$$\sigma^2 \operatorname*{plim}_{n\to\infty} S_{W^TX}^{-1}$$

$$\operatorname*{plim}_{n\to\infty} \frac{\sigma^2}{n}\left(\frac{W^TX}{n}\right)^{-1}\left(\frac{W^TW}{n}\right)\left(\frac{W^TX}{n}\right)^{-1^T}$$

Each is square.

$$\operatorname*{plim}_{n\to\infty} \frac{\sigma^2}{n}\left(\frac{X^TW}{n}\left(\frac{W^TW}{n}\right)^{-1}\frac{W^TX}{n}\right)^{-1}$$

Generalized IV Estimator.

$$\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{u}$$

use instruments, W

$$W^Ty = W^TX\beta + W^T\underset{\sim}{u}$$

$$W^T\underset{\sim}{u} \sim \left(0,\ W^T\sigma^2 I\, W\right)$$
$$\sim \left(0,\ \sigma^2 W^TW\right)$$

use GLS

$$\hat{\beta}_{IV} = \left(X^TW\left(W^TW\right)^{-1}W^TX\right)^{-1}\left(X^TW\left(W^TW\right)^{-1}W^T\underset{\sim}{y}\right)$$

$\text{plim } \hat{\beta}_{IV} = \underset{\sim}{\beta}$

$\text{Cov}(\hat{\beta}_{IV}) = \sigma^2 \left(X^T W (W^T W)^{-1} W^T X \right)^{-1}$

Estimated By

$$\hat{\sigma}^2_{IV} = \frac{(\underset{\sim}{y} - X\hat{\beta}_{IV})^T (y - X\hat{\beta}_{IV})}{n-k}$$

NOTE: The IV Estimate is also a 2SLS estimate.